

Using Tableau to Decide Description Logics with Full Role Negation and Identity

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Abstract

This paper presents a tableau approach for deciding expressive description logics with full role negation and role identity. We consider the description logic $\mathcal{ALBO}^{\text{id}}$, which is the extension of \mathcal{ALC} with the Boolean role operators, inverse of roles, the identity role, and includes full support for individuals and singleton concepts. $\mathcal{ALBO}^{\text{id}}$ is expressively equivalent to the two-variable fragment of first-order logic with equality and subsumes Boolean modal logic. In this paper we define a sound and complete tableau calculus for the $\mathcal{ALBO}^{\text{id}}$ that provides a basis for decision procedures for this logic and all its sublogics. An important novelty of our approach is the use of a generic unrestricted blocking mechanism as a replacement for standard loop checking mechanisms implemented in current description logic systems. Being based on a conceptually simple rule, unrestricted blocking performs case distinctions over whether two individuals are equal or not and equality reasoning to find finite models. The blocking mechanism ties the proof of termination of tableau derivations to the finite model property of $\mathcal{ALBO}^{\text{id}}$.

1 Introduction

Mainstream description logics languages and ontology web languages are equipped with a rich supply of syntactic constructs and operators for supporting the needs of various applications. In these languages the support of concepts and roles, the main syntactic entities in description logic languages, is however rather uneven. For example, in the description logic \mathcal{ALC} (and other popular extensions of \mathcal{ALC}) it is possible to define

a spam filter as a mechanism for filtering out spam emails, and
a sound spam filter as a spam filter that filters out only spam emails,

that is,

(†) $x \in \text{spam-filter}$ iff $x \in \text{mechanism} \wedge \exists y. (x, y) \in \text{filter-out} \wedge y \in \text{spam-email}$, and
 $x \in \text{sound-spam-filter}$ iff $x \in \text{spam-filter} \wedge \forall y. (x, y) \in \text{filter-out} \rightarrow y \in \text{spam-email}$,

by specifying

$$\begin{aligned} \text{spam-filter} &\stackrel{\text{def}}{=} \text{mechanism} \sqcap \exists \text{filter-out.spam-email} \quad \text{and} \\ \text{sound-spam-filter} &\stackrel{\text{def}}{=} \text{spam-filter} \sqcap \neg \exists \text{filter-out.}\neg \text{spam-email}. \end{aligned}$$

But it is not possible to define

a complete spam filter as a spam filter that filters out every spam email,

that is,

$$(\ddagger) \quad x \in \text{complete-spam-filter} \quad \text{iff} \quad x \in \text{spam-filter} \wedge \forall y. y \in \text{spam-email} \rightarrow (x, y) \in \text{filter-out}.$$

This can be expressed by the following.

$$\text{complete-spam-filter} \stackrel{\text{def}}{=} \text{spam-filter} \sqcap \neg \exists \neg \text{filter-out.spam-email}$$

The first occurrence of negation is a concept negation operator, which almost all description logic include, while the second occurrence is a role negation operator. The role negation operator is not available in \mathcal{ALC} or other current description logics that form the basis of OWL DL/1.1/2.0 [Baader et al., 2003, OWL, OWL 2].

The right-hand-sides of both (\dagger) and (\ddagger) involve universal quantification but of a different kind. In (\dagger) it is the image elements of a role that are universally quantified, while in (\ddagger) it is the elements in a concept that are universally quantified. Most description logics can only handle universal quantification of the kind (\dagger) . From an applications-perspective there is however little justification to give preference to one form of universal quantification over the other, since both are useful. (\dagger) expresses the *necessity* of a property (filter out only spam-emails; what is filtered out is necessarily a spam message) and (\ddagger) expresses the *sufficiency* of a property (filter out every spam-email; being a spam message is sufficient to be filtered out). Natural examples of both kinds of universal quantification can be found in many domains and are common in every-day language.

In this paper we are interested in description logics that allow role negation, and can therefore accommodate examples such as the above, but also provide a range of other role operators not typically supported in mainstream description logics. In particular, we focus on a description logic, called $\mathcal{ALBO}^{\text{id}}$. $\mathcal{ALBO}^{\text{id}}$ is the description logic \mathcal{ALC} extended by union of roles, negation of roles, inverse of roles, and the identity role. In addition, it provides full support for ABox individuals and singleton concepts. $\mathcal{ALBO}^{\text{id}}$ is an extension of the description logic \mathcal{ALB} [Hustadt and Schmidt, 2000] with individuals and singleton concepts, called nominals in modal logic, and the identity role. \mathcal{ALB} is the extension of \mathcal{ALC} , in which concepts and roles form Boolean algebras, and additional operators include inverse of roles and a domain restriction operator. It is shown in this paper (Section 2) that the domain restriction operator can be linearly encoded using the other operators of \mathcal{ALB} .

$\mathcal{ALBO}^{\text{id}}$ is a very expressive description logic. It subsumes Boolean modal logic [Gargov and Passy, 1990, Gargov et al., 1987] and tense, hybrid versions of Boolean modal logic with the @ operator and nominals. It can also be shown that $\mathcal{ALBO}^{\text{id}}$ is expressively equivalent to the two-variable fragment of first-order logic with equality. The following constructs and statements can be handled in $\mathcal{ALBO}^{\text{id}}$.

- Role negation, the universal role, the sufficiency or window operator, the image operator, domain and range restriction, cross product, and left and right cylindrification.
- Role identity, role diversity, and the test operator.
- Nominals, the satisfaction operator of hybrid logic and ‘one-of’ concepts of description logics.
- Concept and role inclusions and equivalence axioms in the language of $\mathcal{ALBO}^{\text{id}}$.
- Concept and role assertions in the language of $\mathcal{ALBO}^{\text{id}}$.
- TBox, ABox and RBox statements.
- Boolean combinations of both concept and role inclusions and equivalence axioms. Boolean combinations of concept and role assertions, including negated role assertions, as well as Boolean combinations of any mixture of assertions, inclusion axioms and equivalence axioms.
- Disjoint roles, symmetric roles, serial roles and reflexive roles.

$\mathcal{ALBO}^{\text{id}}$ is in fact very close to the brink of undecidability, because we know that adding (negative occurrences of) role composition to \mathcal{ALB} takes us into undecidable territory [Schmidt and Hustadt, 2003].

Since $\mathcal{ALBO}^{\text{id}}$ subsumes Boolean modal logic it follows from Lutz and Sattler [2002] that the satisfiability problem in $\mathcal{ALBO}^{\text{id}}$ is NExpTime-hard. Grädel et al. [1997] showed that satisfiability in the two-variable first-order fragment with equality is NExpTime-complete. It follows therefore that the computational complexity of $\mathcal{ALBO}^{\text{id}}$ -satisfiability is NExpTime-complete.

None of the current *tableau-based* description logic systems are able to handle $\mathcal{ALBO}^{\text{id}}$, or \mathcal{ALB} , because they do not support full role negation. Description logics with full role negation can be decided however by translation to first-order logic and first-order resolution theorem provers such as MSPASS [Hustadt et al., 1999], SPASS [Weidenbach et al., 2007], E [Schulz, 2002] and VAMPIRE [Riazanov and Voronkov, 1999]. The paper [Hustadt and Schmidt, 2000] shows that the logic \mathcal{ALB} can be decided by translation to first-order logic and ordered resolution. This result is extended by De Nivelle et al. [2000] to \mathcal{ALB} with positive occurrences of composition of roles. $\mathcal{ALBO}^{\text{id}}$ can be embedded into the two-variable fragment of first-order logic with equality which can be decided with first-order resolution methods [De Nivelle and Pratt-Hartmann, 2001]. This means that $\mathcal{ALBO}^{\text{id}}$ is decidable and can be decided using first-order resolution methods.

Few tableau calculi or tableau procedures have been described for description logics with complex role operators, or equivalent dynamic modal logic versions. Ground semantic tableau calculi and tableau decision procedures are presented by De Nivelle et al. [2000] for the modal versions of $\mathcal{ALC}(\sqcup, \sqcap, ^{-1})$, that is, \mathcal{ALC} with role union, role intersection and role inverse. These are extended with the domain restriction operator to $\mathcal{ALC}(\sqcup, \sqcap, ^{-1}, \upharpoonright)$ by Schmidt [2006b]. A semantic tableau decision procedure for \mathcal{ALC} with role intersection, role inverse, role identity and role composition is described by Massacci [2001]. None of these tableaux make provision for the role negation operator however.

A tableau decision procedure for the description logic \mathcal{ALCQIb} which allows for Boolean combinations of ‘safe’ occurrences of negated roles is discussed by Tobies [2001]. From the clausal form of the first-order translation it can be seen that safeness ensures guardedness which is violated by unsafe occurrences of role negation. Guardedness is also important in the definition of a generalisation with relational properties of the dynamic modal logic corresponding to $\mathcal{ALC}(\sqcup, \sqcap, ^{-1})$ in De Nivelle

et al. [2000], where it is shown that hyperresolution and splitting decides this logic. The hyper-resolution decision procedure for this logic is essentially a hypertableau approach with similarities to Motik et al. [2009]. Decidability of propositional dynamic logic with negation of atomic relations using Büchi automata is shown by Lutz and Walther [2004]. Schmidt et al. [2004] presented a sound and complete ground semantic tableau calculus for Peirce logic [Schmidt and Hustadt, 2006], which is equivalent to the extension of \mathcal{ALB} with role composition and role identity. However the tableau is not terminating because reasoning in Peirce logic is not decidable.

In this paper we present a ground semantic tableau approach that decides the description logic $\mathcal{ALBO}^{\text{id}}$. In order to limit the number of individuals in the tableau derivation and guarantee termination we need a mechanism for finding finite models. *Standard loop checking* mechanisms are based on comparing sets of (labelled or unlabelled) concept expressions such as subset blocking or equality blocking in order to detect periodicity in the underlying models. Instead of using the standard loop checking mechanisms, our approach is based on a new inference rule, called the *unrestricted blocking* rule, and equality reasoning. Our approach has the following advantages over standard loop checking.

- It is conceptually simple and easy to implement. It does not require specialised blocking tests and procedural descriptions in terms of status variables. All individuals are blockable and once blocked remain blocked.
- The blocking mechanism is generally sound and complete. This means that it is applicable also to other deduction methods and other logics [Schmidt and Tishkovsky, 2008] including full first-order logic, where it can be used to find finite models.
- It provides greater flexibility in constructing models. For instance, it can be used to construct small models for a satisfiable concept, including domain minimal models.
- The approach has the advantage that it constructs real models, whereas existing tableau procedures for many OWL description logics construct only pseudo-models that are not always real models but can be completed to real models.
- It can be implemented and simulated in first-order logic provers [Baumgartner and Schmidt, 2008].

The style of presentation of our tableau calculus is similar to presentations in modal and hybrid logic, for example [Fitting, 1972, Blackburn, 2000, De Nivelle et al., 2000, Schmidt, 2006b, Schmidt et al., 2004, del Cerro and Gasquet, 2002]. Notable about our calculus compared for example with [Schmidt, 2006b, Schmidt et al., 2004] is that it operates only on ground labelled *concept* expressions. This makes it easier in principle to implement the calculus as extensions of tableau-based description logic systems which can handle singleton concepts and include equality reasoning.

The structure of the paper is as follows. The syntax and semantics of $\mathcal{ALBO}^{\text{id}}$ are defined in Section 2. In Section 3 we prove that $\mathcal{ALBO}^{\text{id}}$ has the effective finite model property by reducing $\mathcal{ALBO}^{\text{id}}$ -satisfiability to the two-variable fragment of the first-order logic. We define a tableau calculus for $\mathcal{ALBO}^{\text{id}}$ in Section 4, and in Section 5, we prove that it is sound and complete without the unrestricted blocking rule. Sections 6 and 7 introduce our blocking mechanism and prove soundness, completeness and termination of the tableau calculus extended with this mechanism. In Section 8 we prove that an appropriate strategy for the application of the unrestricted blocking rule yields a tableau procedure with optimal complexity. Furthermore, we define general criteria

for deterministic decision procedures for $\mathcal{ALBO}^{\text{id}}$ (and any sublogics) in Section 9. Various issues concerning blocking and our results are discussed in Section 10.

The present paper builds on results and techniques presented in two conference papers [Schmidt and Tishkovsky, 2007, 2008], but there are several significant differences. All results of Schmidt and Tishkovsky [2007] are strengthened to the more expressive description logic $\mathcal{ALBO}^{\text{id}}$, rather than \mathcal{ALBO} . The complexity results for tableau reasoning are completely new. New is also the extensive discussion of the approach, and more examples have been included. The present paper contains proofs which Schmidt and Tishkovsky [2007] did not. There is some similarity to the proof sketches in Schmidt and Tishkovsky [2008] but the proofs in the present paper are considerably improved and simplified. An important difference is that the proof of the effective finite model property is no longer based on the conflict elimination technique by Gargov and Passy [1990]. It is obtained by reduction of $\mathcal{ALBO}^{\text{id}}$ to the two-variable fragment of the first-order logic and exploiting the results by Grädel et al. [1997]. Furthermore, Schmidt and Tishkovsky [2008] caters only for \mathcal{ALBO} and not \mathcal{ALBO} with role identity.

2 Syntax and Semantics of $\mathcal{ALBO}^{\text{id}}$

The syntax of $\mathcal{ALBO}^{\text{id}}$ is defined over a signature, denoted by $\sigma = (O, C, R)$, consisting of three disjoint alphabets: O , the alphabet of individuals or object names, C , the alphabet of concept symbols, and R , the alphabet of role symbols. For individuals we use the notation $\ell, \ell', \ell'', \ell_i$, for concept symbols p, p', p_i , and for role symbols r, r_i . The language includes a special role symbol id for the *identity* role which is regarded as a constant. The logical connectives are: \neg (*negation*), \sqcup (*union* for both concepts and roles), \exists (*existential concept restriction*), and $^{-1}$ (*role inverse*). *Concept expressions* (or *concepts*) and *role expressions* (or *roles*) are defined as follows:

$$\begin{aligned} C, D &\stackrel{\text{def}}{=} p \mid \{\ell\} \mid \neg C \mid C \sqcup D \mid \exists R.C, \\ R, S &\stackrel{\text{def}}{=} r \mid R \sqcup S \mid R^{-1} \mid \neg R \mid \text{id}. \end{aligned}$$

p ranges over the set C , ℓ ranges over O , and r ranges over R . We refer to $\{\ell\}$ as a singleton concept. By *length* of an $\mathcal{ALBO}^{\text{id}}$ -expression E we understand the length in symbols of a word which represents E in the above language.

A TBox is a finite set of inclusion statements of the form $C \sqsubseteq D$ and an RBox is a finite set of inclusion statements of the form $R \sqsubseteq S$. An ABox is a finite set of statements of the form $\ell : C$ or $(\ell, \ell') : R$, which are called *concept assertion* or *role assertions*, respectively. A *knowledge base* is a tuple $(\mathcal{T}, \mathcal{R}, \mathcal{A})$ consisting of a TBox \mathcal{T} , an RBox \mathcal{R} , and an ABox \mathcal{A} .

Next, we define the semantics of $\mathcal{ALBO}^{\text{id}}$. A *model* (or an *interpretation*) \mathcal{I} of $\mathcal{ALBO}^{\text{id}}$ is a tuple

$$\mathcal{I} = (\Delta^{\mathcal{I}}, p^{\mathcal{I}}, \dots, \ell^{\mathcal{I}}, \dots, r^{\mathcal{I}}, \dots),$$

where $\Delta^{\mathcal{I}}$ is a non-empty set, and for any concept symbol p , any role symbol r and any individual ℓ :

$$p^{\mathcal{I}} \text{ is a subset of } \Delta^{\mathcal{I}}, \quad \ell^{\mathcal{I}} \in \Delta^{\mathcal{I}} \quad \text{and} \quad r^{\mathcal{I}} \text{ is a binary relation over } \Delta^{\mathcal{I}}.$$

We expand the interpretation \mathcal{I} to all concepts and roles with the use of the following definitions

(for any concepts C, D , roles R, S , and individuals ℓ):

$$\begin{aligned}
\{\ell\}^{\mathcal{I}} &\stackrel{\text{def}}{=} \{\ell^{\mathcal{I}}\}, \\
(\neg C)^{\mathcal{I}} &\stackrel{\text{def}}{=} \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}, \\
(C \sqcup D)^{\mathcal{I}} &\stackrel{\text{def}}{=} C^{\mathcal{I}} \cup D^{\mathcal{I}}, \\
(\exists R.C)^{\mathcal{I}} &\stackrel{\text{def}}{=} \{x \mid \exists y \in C^{\mathcal{I}} (x, y) \in R^{\mathcal{I}}\}, \\
(R \sqcup S)^{\mathcal{I}} &\stackrel{\text{def}}{=} R^{\mathcal{I}} \cup S^{\mathcal{I}}, \\
(R^{-1})^{\mathcal{I}} &\stackrel{\text{def}}{=} (R^{\mathcal{I}})^{-1} = \{(x, y) \mid (y, x) \in R^{\mathcal{I}}\}, \\
(\neg R)^{\mathcal{I}} &\stackrel{\text{def}}{=} (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}) \setminus R^{\mathcal{I}}, \\
\text{id}^{\mathcal{I}} &\stackrel{\text{def}}{=} \{(x, x) \mid x \in \Delta^{\mathcal{I}}\}.
\end{aligned}$$

A concept C is *satisfied* in a model \mathcal{I} iff $C^{\mathcal{I}} \neq \emptyset$, while C is *valid* in a model \mathcal{I} iff $C^{\mathcal{I}} = \Delta^{\mathcal{I}}$. A set of concepts is satisfied (respectively, valid) in a model \mathcal{I} iff all the concepts in the set are satisfied (respectively, valid) in \mathcal{I} . A concept is *satisfiable* iff there is a model where it is satisfied.

If E is a concept expression, a concept inclusion statement, a role inclusion statement, a concept assertion or a role assertion, we indicate by $\mathcal{I} \models E$ that E is valid in the model \mathcal{I} . In particular, shortening the definition of the concept validity, we can write that, in any model \mathcal{I} and for any concept C ,

$$\mathcal{I} \models C \stackrel{\text{def}}{\iff} C^{\mathcal{I}} = \Delta^{\mathcal{I}}.$$

Concept and role inclusion statements are interpreted as subset relationships, and concept and role assertions are interpreted as element-of relationships. In particular, in any model \mathcal{I} and for any concepts C, D , any roles R, S , and any individuals ℓ, ℓ' :

$$\begin{aligned}
\mathcal{I} \models C \sqsubseteq D &\stackrel{\text{def}}{\iff} C^{\mathcal{I}} \subseteq D^{\mathcal{I}}, \\
\mathcal{I} \models R \sqsubseteq S &\stackrel{\text{def}}{\iff} R^{\mathcal{I}} \subseteq S^{\mathcal{I}}, \\
\mathcal{I} \models \ell : C &\stackrel{\text{def}}{\iff} \ell^{\mathcal{I}} \in C^{\mathcal{I}}, \\
\mathcal{I} \models (\ell, \ell') : R &\stackrel{\text{def}}{\iff} (\ell^{\mathcal{I}}, \ell'^{\mathcal{I}}) \in R^{\mathcal{I}}.
\end{aligned}$$

A concept C is *satisfiable with respect to a knowledge base* $(\mathcal{T}, \mathcal{R}, \mathcal{A})$ iff C is satisfied in a model \mathcal{I} validating all the statements from the knowledge base, that is, $\mathcal{I} \models E$ for every $E \in \mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$.

$\mathcal{ALBO}^{\text{id}}$ has considerable expressive power and many useful additional operators can be defined in it. These include:

| | |
|------------------------|--|
| Top concept: | $\top \stackrel{\text{def}}{=} p \sqcup \neg p$ (for some concept symbol p) |
| Bottom concept: | $\perp \stackrel{\text{def}}{=} \neg \top$ |
| Concept intersection: | $C \sqcap D \stackrel{\text{def}}{=} \neg(\neg C \sqcup \neg D)$ |
| Universal restriction: | $\forall R.C \stackrel{\text{def}}{=} \neg \exists R. \neg C$ |
| Image operator: | $\exists^{-1} R.C \stackrel{\text{def}}{=} \exists R^{-1}.C$ |

Additional operators include these operators, which can be defined using role negation.

| | |
|-----------------------|---|
| Top role: | $\nabla \stackrel{\text{def}}{=} r \sqcup \neg r$ (for some role symbol r) |
| Bottom role: | $\Delta \stackrel{\text{def}}{=} \neg \nabla$ |
| Role intersection: | $R \sqcap S \stackrel{\text{def}}{=} \neg(\neg R \sqcup \neg S)$ |
| Diversity role: | $\text{div} \stackrel{\text{def}}{=} \neg \text{id}$ |
| Universal modality: | $\Box C \stackrel{\text{def}}{=} \forall \nabla. C$ |
| Sufficiency operator: | $\bar{\forall} R.C \stackrel{\text{def}}{=} \neg \exists \neg R.C$ |

The sufficiency operator is also known as the window operator.

Concept assertions are defined as concept expressions as follows:

$$\text{Concept assertions:} \quad \ell : C \stackrel{\text{def}}{=} \exists \nabla. (\{\ell\} \sqcap C).$$

A role assertion $(\ell, \ell') : R$ can be expressed as a concept assertion, namely

$$\text{Role assertions:} \quad (\ell, \ell') : R \stackrel{\text{def}}{=} \ell : \exists R. \{\ell'\},$$

or, using the above, as a concept expression. In addition, concept and role inclusion axioms are definable as concept expressions.

$$\begin{aligned} \text{Concept inclusion:} \quad & C \sqsubseteq D \stackrel{\text{def}}{=} \Box(\neg C \sqcup D) \\ \text{Role inclusion:} \quad & R \sqsubseteq S \stackrel{\text{def}}{=} \Box \forall \neg(\neg R \sqcup S). \perp \end{aligned}$$

Concept assertions, role assertions, concept inclusions and role inclusions in $\mathcal{ALBO}^{\text{id}}$ are therefore all representable as concept expressions. This means that in $\mathcal{ALBO}^{\text{id}}$ the distinction between TBoxes, ABoxes and RBoxes is strictly no longer necessary, and concept satisfiability in $\mathcal{ALBO}^{\text{id}}$ with respect to any knowledge base can be reduced to concept satisfiability with respect to a knowledge base where the TBox, the RBox, and the ABox are all empty. We have that a concept C is satisfiable with respect to a knowledge base $(\mathcal{T}, \mathcal{R}, \mathcal{A})$ iff the concept

$$C \sqcap \prod (\mathcal{T} \cup \mathcal{R} \cup \mathcal{A})$$

is satisfiable. Similarly, a set of concepts \mathcal{S} is satisfiable with respect to a knowledge base $(\mathcal{T}, \mathcal{R}, \mathcal{A})$ iff the set $\mathcal{S} \cup \mathcal{T} \cup \mathcal{R} \cup \mathcal{A}$ is satisfiable. Without loss of generality, in this paper, we therefore focus on the problem of concept satisfiability in $\mathcal{ALBO}^{\text{id}}$.

Boolean combinations of inclusion and assertion statements of concepts and roles are expressible in $\mathcal{ALBO}^{\text{id}}$, as the corresponding Boolean combinations of the concept expressions which represent these statements. In fact, we are allowed to use such statements in concept and role expressions within the language of $\mathcal{ALBO}^{\text{id}}$ provided that a well-formed $\mathcal{ALBO}^{\text{id}}$ expression is obtained after replacing all these statements in the original expression for well-formed $\mathcal{ALBO}^{\text{id}}$ expressions which define them. For example, the following expression is a well-formed concept expression in $\mathcal{ALBO}^{\text{id}}$:

$$\left((\text{Alice} : \neg \exists \text{likes.football_fans}) \sqcap (\text{Bob} : \text{football_fans}) \sqcap (\text{hasFriend} \sqsubseteq \text{likes}) \right) \sqsubseteq \left((\text{Alice}, \text{Bob}) : \neg \text{hasFriend} \right).$$

It says that Alice and Bob cannot be friends because Bob is fanatical about football and Alice does not like football fans. The sentence uses an assumption expressed as a role inclusion that somebody's friend is necessarily a person who he or she likes. Notice that the role inclusion statement denotes an $\mathcal{ALBO}^{\text{id}}$ expression which essentially uses the role negation operator.

$\mathcal{ALBO}^{\text{id}}$ also has enough expressive power to represent the operators defined below:

$$\begin{array}{ll}
\text{Test operator:} & (C?)^{\mathcal{I}} \stackrel{\text{def}}{=} \{(x, x) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid x \in C^{\mathcal{I}}\}, \\
\text{Domain restriction:} & (R\upharpoonright C)^{\mathcal{I}} \stackrel{\text{def}}{=} \{(x, y) \in R \mid x \in C^{\mathcal{I}}\}, \\
\text{Range restriction:} & (R\downharpoonright C)^{\mathcal{I}} \stackrel{\text{def}}{=} \{(x, y) \in R \mid y \in C^{\mathcal{I}}\}, \\
\text{Left cylindrification:} & (D^c)^{\mathcal{I}} \stackrel{\text{def}}{=} \{(x, y) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid x \in D^{\mathcal{I}}\}, \\
\text{Right cylindrification:} & ({}^cD)^{\mathcal{I}} \stackrel{\text{def}}{=} \{(x, y) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid y \in D^{\mathcal{I}}\}, \\
\text{Cross product:} & (C \times D)^{\mathcal{I}} \stackrel{\text{def}}{=} C^{\mathcal{I}} \times D^{\mathcal{I}} = \{(x, y) \mid x \in C^{\mathcal{I}}, y \in D^{\mathcal{I}}\}.
\end{array}$$

The test operator can be represented using the range restriction operator and the identity role:

$$(C?)^{\mathcal{I}} = (\text{id}\downharpoonright C)^{\mathcal{I}}.$$

In the presence of role inverse, the top role and role intersection any single operator from $\{\upharpoonright, \downharpoonright, \cdot^c, {}^c, \times\}$ is enough to define the remaining operators from this set. In any model \mathcal{I} , the following equalities are true:

$$\begin{array}{ll}
(R\upharpoonright C)^{\mathcal{I}} = ((R^{-1}\downharpoonright C)^{-1})^{\mathcal{I}} & (R\downharpoonright C)^{\mathcal{I}} = ((R^{-1}\upharpoonright C)^{-1})^{\mathcal{I}} \\
(D^c)^{\mathcal{I}} = (({}^cD)^{-1})^{\mathcal{I}} & ({}^cD)^{\mathcal{I}} = (D^{c^{-1}})^{\mathcal{I}} \\
(D^c)^{\mathcal{I}} = (\nabla\downharpoonright D)^{\mathcal{I}} & ({}^cD)^{\mathcal{I}} = (\nabla\upharpoonright D)^{\mathcal{I}} \\
(R\downharpoonright D)^{\mathcal{I}} = (R \sqcap D^c)^{\mathcal{I}} & (R\upharpoonright D)^{\mathcal{I}} = (R \sqcap {}^cD)^{\mathcal{I}} \\
(D^c)^{\mathcal{I}} = (D \times \top)^{\mathcal{I}} & ({}^cD)^{\mathcal{I}} = (\top \times D)^{\mathcal{I}} \\
(C \times D)^{\mathcal{I}} = (C^c \sqcap {}^cD)^{\mathcal{I}} &
\end{array}$$

Now it is enough to show that one of these operators can be encoded in $\mathcal{ALBO}^{\text{id}}$. For example, expressions involving left cylindrification can be *linearly* encoded in $\mathcal{ALBO}^{\text{id}}$ by replacing all occurrences of D^c in C by a new role symbol r_D uniquely associated with D and adding the definitions

$$\neg D \sqsubseteq \forall r_D. \perp \quad \text{and} \quad D \sqsubseteq \neg \exists \neg r_D. \top$$

to the knowledge base. This encoding preserves satisfiability equivalence. In a similar way, the other operators from $\{\upharpoonright, \downharpoonright, \cdot^c, {}^c, \times\}$ can be linearly encoded in $\mathcal{ALBO}^{\text{id}}$.

Often description logics are required to satisfy the unique name assumption. We do not assume it for $\mathcal{ALBO}^{\text{id}}$. On the other hand, assuming it would be inconsequential, because the unique name assumption can be enforced by disjointness statements of the form $\{\ell\} \sqcap \{\ell'\} \sqsubseteq \perp$, for every distinct pair of individuals that occur in the given knowledge base.

In this paper we do not use role assertions in the form $(\ell, \ell') : R$ but use them in the form $\ell : \exists R. \{\ell'\}$. We refer to $\ell : \exists R. \{\ell'\}$ as a *link* (between the individuals ℓ and ℓ').

Later we refer to the description logics \mathcal{ALCO} and \mathcal{ALCO}^{\square} . \mathcal{ALCO} is a sublogic of $\mathcal{ALBO}^{\text{id}}$ without all the role operators and the identity role. \mathcal{ALCO}^{\square} extends \mathcal{ALCO} with the universal modality operator \square .

3 Effective Finite Model Property

In this section we show that $\mathcal{ALBO}^{\text{id}}$ has the effective finite model property, encoding $\mathcal{ALBO}^{\text{id}}$ -satisfiability of concepts in the two-variable fragment of the first-order logic with equality.

For every concept symbol p , role symbol r , and individual ℓ , let \mathbf{p} be a unary predicate symbol, \mathbf{r} a binary predicate symbol, and ℓ a constant which are uniquely associated with p , r , and ℓ , respectively. The following defines a mapping ST as the standard translation of $\mathcal{ALBO}^{\text{id}}$ parametrised by the variables x and y . The standard translation of concepts is:

$$\begin{aligned} \text{ST}_x(p) &\stackrel{\text{def}}{=} \mathbf{p}(x) & \text{ST}_x(\{\ell\}) &\stackrel{\text{def}}{=} x \approx \ell \\ \text{ST}_x(\neg C) &\stackrel{\text{def}}{=} \neg \text{ST}_x(C) & \text{ST}_x(C \sqcup D) &\stackrel{\text{def}}{=} \text{ST}_x(C) \vee \text{ST}_x(D) \\ \text{ST}_x(\exists R.C) &\stackrel{\text{def}}{=} \exists y (\text{ST}_{xy}(R) \wedge \text{ST}_y(C)) \end{aligned}$$

The standard translation of roles is:

$$\begin{aligned} \text{ST}_{xy}(r) &\stackrel{\text{def}}{=} \mathbf{r}(x, y) & \text{ST}_{xy}(\text{id}) &\stackrel{\text{def}}{=} x \approx y \\ \text{ST}_{xy}(R^{-1}) &\stackrel{\text{def}}{=} \text{ST}_{yx}(R) & \text{ST}_{xy}(R \sqcup S) &\stackrel{\text{def}}{=} \text{ST}_{xy}(R) \vee \text{ST}_{xy}(S) \\ \text{ST}_{xy}(\neg R) &\stackrel{\text{def}}{=} \neg \text{ST}_{xy}(R) \end{aligned}$$

Clearly, only two variables occur in $\text{ST}_x(C)$ for every concept C . Given a concept C , the translation $\text{ST}_x(C)$ of C can be computed in linear time with respect to the length of C .

As the standard translation just follows the definition of the semantics of $\mathcal{ALBO}^{\text{id}}$, every $\mathcal{ALBO}^{\text{id}}$ -model \mathcal{I} can be viewed as a first-order model over the signature $\sigma = (\{\ell \mid \ell \in O\}, \{\mathbf{p} \mid p \in C\}, \{\mathbf{r} \mid r \in R\})$, where the interpretations of the symbols \mathbf{p} , \mathbf{r} , and ℓ are $p^{\mathcal{I}}$, $r^{\mathcal{I}}$, and $\ell^{\mathcal{I}}$ respectively. Similarly, every first-order model over a signature σ can be seen as an $\mathcal{ALBO}^{\text{id}}$ -model.

We write $\mathcal{I} \models \phi[x_1 \mapsto a_1, \dots, x_n \mapsto a_n]$ to indicate that all free variables of a first-order formula ϕ are contained in the set $\{x_1, \dots, x_n\}$ and ϕ is true in \mathcal{I} where the variables x_1, \dots, x_n are assigned the domain elements a_1, \dots, a_n , respectively.

The following theorem can be proved by induction on the length of a concept.

Theorem 1. *For every concept C , every $\mathcal{ALBO}^{\text{id}}$ -model \mathcal{I} and $a \in \Delta^{\mathcal{I}}$, we have $a \in C^{\mathcal{I}}$ iff $\mathcal{I} \models \text{ST}_x(C)[x \mapsto a]$.*

The effective finite model property for description logics can be stated as follows. A description logic L has the *effective finite model property* iff there is a function $\mu : \mathbb{N} \rightarrow \mathbb{N}$, such that the following holds.

For every concept C , if C is satisfiable in an L -model then there is a finite L -model for C with the number of elements in the domain not exceeding $\mu(n)$, where n is the length of C .

We call μ the (*model*) *bounding function* for L .

The notion of the effective finite model property for fragments of the first-order logic can be obtained from the above definition replacing phrases “logic L ” for “fragment \mathcal{F} ” and “concept C ” for “formula ϕ ”.

The effective finite model property is sometimes called the bounded model property or the small model property and is stronger than the finite model property. A logic has the finite model property

if any satisfiable concept (formula) has a finite model. With the effective finite model property there is a bound on the size of the model that is given a priori based on the given concept (or formula). This means there is a naive procedure for deciding satisfiability, which is based on successively computing larger and larger models within the bound and model checking the given concept with respect to the obtained models.

The effective finite model property for the two-variable fragment of the first-order logic with equality is proved by Mortimer [1975]. An exponential model bounding function μ is given by Grädel et al. [1997].

Theorem 2 (Grädel et al. 1997). *The two variable fragment of the first order logic with equality has the effective finite model property with the model bounding function $\mu(n) = 3(n \lfloor \log(n+1) \rfloor) \cdot 2^n$.*

Theorems 1 and 2 imply that $\mathcal{ALBO}^{\text{id}}$ has the effective finite model property:

Theorem 3 (Effective Finite Model Property of $\mathcal{ALBO}^{\text{id}}$). *$\mathcal{ALBO}^{\text{id}}$ has the effective finite model property with the model bounding function $\mu(n) = 3(n \lfloor \log(n+1) \rfloor) \cdot 2^n$.*

This provides an upper bound for the worst-case complexity of testing concept satisfiability for $\mathcal{ALBO}^{\text{id}}$. Combining this with the complexity result for Boolean modal logic [Lutz and Sattler, 2002], it follows that:

Theorem 4. *The concept satisfiability problem for $\mathcal{ALBO}^{\text{id}}$ is NExpTime-complete.*

4 Tableau Calculus

Let T denote a tableau calculus comprising of a set of inference rules. A *derivation* or *tableau* for T is a finitely branching, ordered tree whose nodes are sets of labelled concept expressions. Assuming that \mathcal{S} is the input set of concept expressions to be tested for satisfiability the root node of the tableau is the set $\{\ell : C \mid C \in \mathcal{S}\}$, where ℓ denotes a fresh individual. Successor nodes are constructed in accordance with a set of inference rules in the calculus. The inference rules have the general form

$$\frac{X_0}{X_1 \mid \dots \mid X_n},$$

where X_0 is the set of premises and the X_i are the sets of conclusions. If $n = 0$, the rule is called *closure rule* and written X_0/\perp . An inference rule is applicable to a selected labelled concept expression E in a node of the tableau, if E together with possibly other labelled concept expressions in the node, are simultaneous instantiations of all the premises of the rule. Then n successor nodes are created which contain the formulae of the current node and the appropriate instances of X_1, \dots, X_n . We assume that *any rule is applied at most once to the same set of premises*, which is a standard assumption for tableau derivations.

We use the notation $T(\mathcal{S})$ for a fully expanded tableau built by applying the rules of the calculus T starting with the set \mathcal{S} as input. That is, we assume that all branches in the tableau are fully expanded and all applicable rules have been applied in $T(\mathcal{S})$. As usual we assume that all the rules of the calculus are applied *non-deterministically to a tableau*. For any concepts C_1, \dots, C_n , the notation $T(C_1, \dots, C_n)$ is used instead of $T(\{C_1, \dots, C_n\})$.

In a tableau, a maximal path from the root node is called a *branch*. For \mathcal{B} a branch of a tableau we write $D \in \mathcal{B}$ to indicate that the concept D has been derived in \mathcal{B} , that is, D belongs to a node

Rules for \mathcal{ALCCO} :

$$\begin{array}{ll}
(\perp): \frac{\ell : C, \ell : \neg C}{\perp} & (\neg\neg): \frac{\ell : \neg\neg C}{\ell : C} \\
(\neg\sqcup): \frac{\ell : \neg(C \sqcup D)}{\ell : \neg C, \ell : \neg D} & (\sqcup): \frac{\ell : (C \sqcup D)}{\ell : C \mid \ell : D} \\
(\exists): \frac{\ell : \exists R.C}{\ell : \exists R.\{\ell'\}, \ell' : C} \text{ (\ell' is new)} & (\neg\exists): \frac{\ell : \neg\exists R.C, \ell : \exists R.\{\ell'\}}{\ell' : \neg C} \\
(\text{sym}): \frac{\ell : \{\ell'\}}{\ell' : \{\ell\}} & (\neg\text{sym}): \frac{\ell : \neg\{\ell'\}}{\ell' : \neg\{\ell\}} \\
(\text{mon}): \frac{\ell : \{\ell'\}, \ell' : C}{\ell : C} & (\text{refl}): \frac{\ell : C}{\ell : \{\ell\}}
\end{array}$$

Rules for complex roles:

$$\begin{array}{ll}
(\exists\sqcup): \frac{\ell : \exists(R \sqcup S).\{\ell'\}}{\ell : \exists R.\{\ell'\} \mid \ell : \exists S.\{\ell'\}} & (\neg\exists\sqcup): \frac{\ell : \neg\exists(R \sqcup S).C}{\ell : \neg\exists R.C, \ell : \neg\exists S.C} \\
(\exists^{-1}): \frac{\ell : \exists R^{-1}.\{\ell'\}}{\ell' : \exists R.\{\ell\}} & (\neg\exists^{-1}): \frac{\ell : \neg\exists R^{-1}.C, \ell' : \exists R.\{\ell\}}{\ell' : \neg C} \\
(\exists\neg): \frac{\ell : \exists\neg R.\{\ell'\}}{\ell : \neg\exists R.\{\ell'\}} & (\neg\exists\neg): \frac{\ell : \neg\exists\neg R.C, \ell' : \{\ell'\}}{\ell : \exists R.\{\ell'\} \mid \ell' : \neg C} \\
(\exists\text{id}): \frac{\ell : \exists\text{id}.\{\ell'\}}{\ell : \{\ell'\}} & (\neg\exists\text{id}): \frac{\ell : \neg\exists\text{id}.C}{\ell : \neg C}
\end{array}$$

Figure 1: Tableau calculus $T_{\mathcal{ALBO}^{\text{id}}}$ for $\mathcal{ALBO}^{\text{id}}$.

of the branch \mathcal{B} . The notion of a tableau branch we use in this paper can be viewed in two ways. On one hand it has a procedural flavour as a path of nodes in the tableau, and on the other hand a branch can be identified with the set-theoretical union of the nodes in it. A more careful distinction between these perspectives leads to the classical notion of a Hintikka set but this is not essential for the paper.

A branch of a tableau is *closed* if a closure rule has been applied in this branch, otherwise the branch is called *open*. The tableau $T(\mathcal{S})$ is *closed* if all its branches are closed and $T(\mathcal{S})$ is *open* otherwise. The calculus T is *sound* iff for any (possibly infinite) set of concepts \mathcal{S} , each $T(\mathcal{S})$ is open whenever \mathcal{S} is satisfiable. T is *complete* iff for any (possibly infinite) unsatisfiable set of concepts \mathcal{S} there is a $T(\mathcal{S})$ which is closed. T is said to be *terminating* (for satisfiability) iff for every *finite* set of concepts \mathcal{S} every closed tableau $T(\mathcal{S})$ is finite and every open tableau $T(\mathcal{S})$ has a finite open branch.

Let $T_{\mathcal{ALBO}^{\text{id}}}$ be the tableau calculus consisting of the rules listed in Figure 1.

The first group of rules are standard for testing satisfiability of concepts for the description logic \mathcal{ALCCO} , that is, \mathcal{ALC} with individuals. The (\perp) rule is the closure rule. The $(\neg\neg)$ rule

removes occurrences of double negation on concepts. (The rule is superfluous if double negations are eliminated using on-the-fly rewrite rules, but is included to simplify the completeness proof somewhat.) The (\sqcup) and $(\neg\sqcup)$ rules are standard rules for handling concept disjunctions. As usual, and in accordance with the semantics of the existential restriction operator, for any existentially restricted concept the (\exists) rule creates a new individual with this concept and adds a link to the new individual. It is the only rule in the calculus that generates new individuals. An additional side-condition is that the new individual ℓ' is uniquely associated with the premise $\ell : \exists R.C$. As usual we assume that the (\exists) rule is applied only to expressions of the form $\ell : \exists R.C$, when C is not a singleton, that is, $C \neq \{\ell''\}$ for any individual ℓ'' . This condition prevents the application of the (\exists) rule to expressions of the form $\ell : \exists R.\{\ell''\}$ because this would just cause the creation of another witness of an R -successor of ℓ when ℓ'' is already such a witness. The $(\neg\exists)$ rule is equivalent to the standard rule for universally restricted concept expressions. The (sym), (mon), and (refl) rules are the equality rules for individuals, familiar from hybrid logic tableau systems. They can be viewed as versions of standard rules for first-order equality. The (refl) rule is formulated perhaps a bit unusually, but the purpose of the premise is to ensure that the rule is realised only for individuals actually occurring in the branch. The $(\neg\text{sym})$ rule is needed to ensure that any negated singleton concept eventually appears as a label in a concept assertion.

The rules in the second group are the rules for decomposing complex role expressions. They can be divided into two subgroups: rules for positive existential role occurrences and rules for negated existential role occurrences. These are listed in the left and right columns, respectively. Due to the presence of the (\exists) rule, the rules for positive existential roles are restricted to role assertions, that is, the concept expressions immediately below the \exists operator are singleton concepts.

Among the rules for negated existential roles, the $(\neg\exists^{-1})$ rule and the $(\neg\exists\neg)$ rule are special. The $(\neg\exists^{-1})$ rule allows the backward propagation of concept expressions along inverted links (ancestor links). The $(\neg\exists\neg)$ rule is the rule for the sufficiency operator. It expands a universally restricted concept in which the role is negated according to the semantics:

$$x \in (\neg\exists\neg R.C)^{\mathcal{I}} \iff \forall y ((x, y) \in R^{\mathcal{I}} \vee y \in (\neg C)^{\mathcal{I}}).$$

That is, ℓ' in the rule is implicitly quantified by a universal quantifier. The effect of the second premise, $\ell' : \{\ell'\}$, is to instantiate ℓ' with individuals that occur in the branch. The remaining rules in this subgroup are based on obvious logical equivalences in $\mathcal{ALBO}^{\text{id}}$.

Tableau rules that do not produce new individuals are called *type-completing* rules. In the case of the calculus $T_{\mathcal{ALBO}^{\text{id}}}$, with the exception of the (\exists) rule, all rules are type-completing.

Now, given an input set of concepts \mathcal{S} , a tableau derivation is constructed as follows. First, preprocessing is performed. This pushes occurrences of the role inverse operator in every concept in \mathcal{S} inward toward atomic concepts by exhaustively applying the following role equivalences from left to right:

$$\begin{aligned} (R \sqcup S)^{-1} &= R^{-1} \sqcup S^{-1}, \\ (R^{-1})^{-1} &= R, \\ (\neg R)^{-1} &= \neg(R^{-1}), \\ \text{id}^{-1} &= \text{id}. \end{aligned}$$

This preprocessing is required to allow the tableau algorithm to handle the role inverse operator correctly. Pushing inverse inwards is the only preprocessing required and can be done in linear

1. $\ell_0 : \neg(\neg\exists(r \sqcup \neg r).p \sqcup \exists r.p)$ given
2. $\ell_0 : \neg\neg\exists(r \sqcup \neg r).p$ $(\neg\sqcup),1$
3. $\ell_0 : \neg\exists r.p$ $(\neg\sqcup),1$
4. $\ell_0 : \exists(r \sqcup \neg r).p$ $(\neg\neg),2$
5. $\ell_0 : \exists(r \sqcup \neg r).\{\ell_1\}$ $(\exists),4$
6. $\ell_1 : p$ $(\exists),4$
7. $\ell_0 : \{\ell_0\}$ (refl),1
8. $\ell_1 : \{\ell_1\}$ (refl),6
9. ▶ $\ell_0 : \exists r.\{\ell_1\}$ $(\exists\sqcup),5$
10. $\ell_1 : \neg p$ $(\neg\exists),3$
11. Unsatisfiable $(\perp),6,10$
12. ▶ $\ell_0 : \exists\neg r.\{\ell_1\}$ $(\exists\sqcup),5$
13. $\ell_0 : \neg\exists r.\{\ell_1\}$ $(\exists\sqcup),5$
14. Satisfiable (no more rules are applicable)

Figure 2: A derivation in $T_{\mathcal{ALBO}^{\text{id}}}$

time. Transformation to negation normal form is optional but for practical purposes not a good choice because obvious properties of expressions and their negation require unnecessary inference steps to reveal.

Suppose \mathcal{S}' is the set of concepts resulting from preprocessing. Then we build a complete tableau $T_{\mathcal{ALBO}^{\text{id}}}(\mathcal{S}')$ as described above.

An example of a finite derivation in $T_{\mathcal{ALBO}^{\text{id}}}$ is shown on Figure 2 for the concept

$$\neg(\neg\exists(r \sqcup \neg r).p \sqcup \exists r.p).$$

In the figure each line in the derivation is numbered on the left. The rule applied and the number of the premise(s) to which it was applied to produce the labelled concept expression (assertion) in each line is specified on the right. The black triangles denote branching points in the derivation. A branch expansion after a branching point is indicated by appropriate indentation. The derivation presents a finished tableau with two branches; the left branch is closed and the right branch is open. Hence, because the tableau calculus is complete, which is shown in the next section, the input concept $\neg(\neg\exists(r \sqcup \neg r).p \sqcup \exists r.p)$ is satisfiable.

In the rest of the paper when we refer to the calculus $T_{\mathcal{ALBO}^{\text{id}}}$ we assume that the described preprocessing with respect to role inverse has been applied to the input concept (set) before the rules of the calculus are applied. It is also important to note that $\ell : C$ and all labelled expressions and assertions really denote concept expressions.

5 Soundness and Completeness

We turn to proving soundness and completeness of the calculus. Because every rule preserves the satisfiability of concept assertions, it is easy to see that the calculus $T_{\mathcal{ALBO}^{\text{id}}}$ is sound for $\mathcal{ALBO}^{\text{id}}$.

Theorem 5 (Soundness). *$T_{\mathcal{ALBO}^{\text{id}}}$ is a sound tableau calculus for satisfiability in $\mathcal{ALBO}^{\text{id}}$.*

For proving completeness, suppose that a tableau $T_{\mathcal{ALBO}^{\text{id}}}(\mathcal{S})$ for a given set of concepts \mathcal{S} is open, that is, it contains an open branch \mathcal{B} . From \mathcal{B} we construct a model $\mathcal{I}(\mathcal{B})$ for the satisfiability of \mathcal{S} as follows. By definition, let

$$\ell \sim_{\mathcal{B}} \ell' \stackrel{\text{def}}{\iff} \ell : \{\ell'\} \in \mathcal{B}.$$

The rules (sym), (mon), and (refl) ensure that $\sim_{\mathcal{B}}$ is an equivalence relation on individuals. Define the equivalence class $\|\ell\|$ of a representative ℓ by:

$$\|\ell\| \stackrel{\text{def}}{=} \{\ell' \mid \ell \sim_{\mathcal{B}} \ell'\}.$$

We set

$$\begin{aligned} \Delta^{\mathcal{I}(\mathcal{B})} &\stackrel{\text{def}}{=} \{\|\ell\| \mid \ell : \{\ell\} \in \mathcal{B}\}, \\ \text{id}^{\mathcal{I}(\mathcal{B})} &\stackrel{\text{def}}{=} \{(\|\ell\|, \|\ell\|) \mid \ell : \{\ell\} \in \mathcal{B}\}, \end{aligned}$$

and for every $r \in R$, $p \in C$ and $\ell \in O$ we define

$$\begin{aligned} (\|\ell\|, \|\ell'\|) \in r^{\mathcal{I}(\mathcal{B})} &\stackrel{\text{def}}{\iff} \ell : \exists r.\{\ell''\} \in \mathcal{B} \text{ for some } \ell'' \sim_{\mathcal{B}} \ell', \\ \|\ell\| \in p^{\mathcal{I}(\mathcal{B})} &\stackrel{\text{def}}{\iff} \ell : p \in \mathcal{B}, \\ \ell^{\mathcal{I}(\mathcal{B})} &\stackrel{\text{def}}{=} \begin{cases} \|\ell\|, & \text{if } \ell : \{\ell\} \in \mathcal{B}, \\ \|\ell'\| & \text{for some } \|\ell'\| \in \Delta^{\mathcal{I}(\mathcal{B})}, \text{ otherwise.} \end{cases} \end{aligned}$$

The rule (mon) ensures that the definitions of $r^{\mathcal{I}(\mathcal{B})}$ and $p^{\mathcal{I}(\mathcal{B})}$ are correct and do not depend on the choice of the representative of the equivalence class $\|\ell\|$. Finally, we expand the interpretation $\mathcal{I}(\mathcal{B})$ to all concepts and roles in the expected way using induction on their lengths. This completes the definition of the model $\mathcal{I}(\mathcal{B})$ extracted from a branch \mathcal{B} .

Let \prec be the ordering on expressions of $\mathcal{ALBO}^{\text{id}}$ induced by the rules of $T_{\mathcal{ALBO}^{\text{id}}}$. That is, \prec is the smallest transitive ordering on the set of all $\mathcal{ALBO}^{\text{id}}$ expressions (concepts and roles) satisfying:

$$\begin{array}{ll} C \prec \neg C & R \prec \neg R \\ C \prec C \sqcup D & D \prec C \sqcup D \\ \neg C \prec \neg(C \sqcup D) & \neg D \prec \neg(C \sqcup D) \\ C \prec \exists R.C & R \prec \exists R.C \\ \neg C \prec \neg \exists R.C & R \prec \neg \exists R.C \\ R \prec R \sqcup S & S \prec R \sqcup S \\ R \prec R^{-1} & \end{array}$$

Note that \prec does not coincide with the direct subexpression ordering. \prec is a well-founded ordering, since the length of the expression to the left in a clause is always strictly less than the length of the expression to the right.

Lemma 6. (1) If $\ell : D \in \mathcal{B}$ then $\|\ell\| \in D^{\mathcal{I}(\mathcal{B})}$ for any concept D .

(2) For every role R and every concept D

(2a) $\ell : \exists R.\{\ell'\} \in \mathcal{B}$ implies $(\|\ell\|, \|\ell'\|) \in R^{\mathcal{I}(\mathcal{B})}$,

(2b) if $(\|\ell\|, \|\ell'\|) \in R^{\mathcal{I}(\mathcal{B})}$ and $\ell : \neg\exists R.D \in \mathcal{B}$ then $\ell' : \neg D \in \mathcal{B}$.

Proof. We prove both properties simultaneously by induction on the ordering \prec . The induction hypothesis is: for an arbitrary $\mathcal{ALBO}^{\text{id}}$ expression E and each expression F such that $F \prec E$, if F is a concept then property (1) holds with $D = F$. Otherwise (that is, if F is a role), property (2) holds with $R = F$ (and D is arbitrary).

To prove property (1) we consider the following cases.

$D = p$. We have $\ell : p \in \mathcal{B} \iff \|\ell\| \in p^{\mathcal{I}(\mathcal{B})}$ by definition of $p^{\mathcal{I}(\mathcal{B})}$.

$D = \{\ell'\}$. If $\ell : \{\ell'\} \in \mathcal{B}$ then, using the definition of $\ell^{\mathcal{I}(\mathcal{B})}$, $\ell^{\mathcal{I}(\mathcal{B})} = \|\ell\| = \|\ell'\| = \ell'^{\mathcal{I}(\mathcal{B})}$ and, consequently, $\ell^{\mathcal{I}(\mathcal{B})} \in \{\ell'^{\mathcal{I}(\mathcal{B})}\} = \{\ell'\}^{\mathcal{I}(\mathcal{B})}$.

$D = \neg D_0$. If $\ell : \neg D_0 \in \mathcal{B}$ then $\ell : D_0 \notin \mathcal{B}$ because otherwise \mathcal{B} would have been closed by the (\perp) rule. We have the following subcases.

$D_0 = p$. We have $\ell : p \in \mathcal{B} \iff \|\ell\| \in p^{\mathcal{I}(\mathcal{B})}$ by definition of $p^{\mathcal{I}(\mathcal{B})}$.

$D_0 = \{\ell'\}$. As the rules $(\neg\text{sym})$ and (refl) have been applied in \mathcal{B} it is clear that $\ell' : \{\ell'\}$ is in \mathcal{B} and $\ell'^{\mathcal{I}(\mathcal{B})} = \|\ell'\|$. Similarly, $\ell^{\mathcal{I}(\mathcal{B})} = \|\ell\|$. Furthermore, because $\ell : \{\ell'\} \notin \mathcal{B}$ we have $\ell \not\sim_{\mathcal{B}} \ell'$, that is $\ell \notin \|\ell'\| = \ell'^{\mathcal{I}(\mathcal{B})}$. Consequently, $\|\ell\| \notin \{\|\ell'\|\} = \{\ell'\}^{\mathcal{I}(\mathcal{B})}$.

$D_0 = \neg D_1$. If $\ell : \neg\neg D_1 \in \mathcal{B}$ then $\ell : D_1 \in \mathcal{B}$ by the $(\neg\neg)$ rule. By the induction hypothesis, the property (1) holds for $D_1 \prec \neg\neg D_1$. Thus, $\|\ell\| \in D_1^{\mathcal{I}(\mathcal{B})} = (\neg\neg D_1)^{\mathcal{I}(\mathcal{B})} = D^{\mathcal{I}(\mathcal{B})}$.

$D_0 = D_1 \sqcup D_2$. In this case both $\ell : \neg D_1$ and $\ell : \neg D_2$ are in the branch \mathcal{B} by the $(\neg\sqcup)$ rule. We also have $\neg D_1 \prec \neg(D_1 \sqcup D_2)$ and $\neg D_2 \prec \neg(D_1 \sqcup D_2)$. Hence, by the induction hypothesis for the property (1), $\|\ell\| \in (\neg D_1)^{\mathcal{I}(\mathcal{B})} \cap (\neg D_2)^{\mathcal{I}(\mathcal{B})} = (\neg(D_1 \sqcup D_2))^{\mathcal{I}(\mathcal{B})}$.

$D_0 = \exists R.D_1$. Let $\|\ell'\|$ be an arbitrary element of $\Delta^{\mathcal{I}(\mathcal{B})}$ such that $(\|\ell\|, \|\ell'\|) \in R^{\mathcal{I}(\mathcal{B})}$ (if there is no such element then there is nothing to prove). By the induction hypothesis the property (2b) holds for $R \prec D_0$. Thus, $\ell' : \neg D_1 \in \mathcal{B}$. The induction hypothesis for the property (1) gives us $\|\ell'\| \notin D_1^{\mathcal{I}(\mathcal{B})}$. Finally, we obtain $\|\ell\| \in (\neg\exists R.D_1)^{\mathcal{I}(\mathcal{B})}$ because ℓ' was chosen arbitrarily.

$D = D_0 \sqcup D_1$. If $\ell : D_0 \sqcup D_1 \in \mathcal{B}$ then either $\ell : D_0 \in \mathcal{B}$, or $\ell : D_1 \in \mathcal{B}$ by the (\sqcup) rule. Hence, either $\|\ell\| \in D_0^{\mathcal{I}(\mathcal{B})}$ or $\|\ell\| \in D_1^{\mathcal{I}(\mathcal{B})}$ by the induction hypothesis for $D_0 \prec D$ and $D_1 \prec D$. Thus, $\|\ell\| \in D_0^{\mathcal{I}(\mathcal{B})} \cup D_1^{\mathcal{I}(\mathcal{B})} = D^{\mathcal{I}(\mathcal{B})}$.

$D = \exists R.D_0$. If $\ell : \exists R.D_0 \in \mathcal{B}$ then $\ell' : D_0 \in \mathcal{B}$ and $\ell : \exists R.\{\ell'\} \in \mathcal{B}$ for some individual ℓ' by the (\exists) rule. By the induction hypothesis the properties (1) and (2a) hold for $D_0 \prec D$ and $R \prec D$ respectively. Hence, $\|\ell'\| \in D_0^{\mathcal{I}(\mathcal{B})}$ and $(\|\ell\|, \|\ell'\|) \in R^{\mathcal{I}(\mathcal{B})}$. That is, $\|\ell\| \in (\exists R.D_0)^{\mathcal{I}(\mathcal{B})}$.

To prove property (2) we consider all cases corresponding to the possible forms that a role R can have.

$R = r$. Since $\ell : \exists r.\{\ell'\} \in \mathcal{B}$ implies $(\|\ell\|, \|\ell'\|) \in r^{\mathcal{I}(\mathcal{B})}$ using the definition of $r^{\mathcal{I}(\mathcal{B})}$, property (2a) holds trivially. For (2b) let $(\|\ell\|, \|\ell'\|) \in r^{\mathcal{I}(\mathcal{B})}$ and $\ell : \neg\exists r.D \in \mathcal{B}$. Using the definition of $r^{\mathcal{I}(\mathcal{B})}$, there is an $\ell'' \sim_{\mathcal{B}} \ell'$ such that $\ell : \exists r.\{\ell''\} \in \mathcal{B}$. Therefore, by the $(\neg\exists)$ rule, $\ell'' : \neg D$ is in the branch \mathcal{B} , too. Finally, $\ell' : \neg D \in \mathcal{B}$ by the (mon) rule.

$R = \text{id}$. For property (2a) let $\ell : \exists \text{id}.\{\ell'\} \in \mathcal{B}$. Then $\ell : \{\ell'\} \in \mathcal{B}$ by the $(\exists \text{id})$ rule and, hence, $\|\ell\| = \|\ell'\|$. Consequently, $(\|\ell\|, \|\ell'\|) \in \text{id}^{\mathcal{I}(\mathcal{B})}$. For (2b) suppose that $(\|\ell\|, \|\ell'\|) \in \text{id}^{\mathcal{I}(\mathcal{B})}$ and $\ell : \neg\exists \text{id}.D \in \mathcal{B}$. Hence, $\|\ell\| = \|\ell'\|$ and, therefore, $\ell : \{\ell'\}$ is in \mathcal{B} . Further, by the $(\neg\exists \text{id})$ rule, $\ell : \neg D \in \mathcal{B}$. Finally, by the (mon) rule, $\ell' : \neg D$ is in the branch \mathcal{B} .

$R = S^{-1}$. For property (2a) let $\ell : \exists S^{-1}.\{\ell'\} \in \mathcal{B}$. Then $\ell' : \exists S.\{\ell\} \in \mathcal{B}$ by the rule (\exists^{-1}) . By the induction hypothesis for $S \prec R$ we have $(\|\ell'\|, \|\ell\|) \in S^{\mathcal{I}(\mathcal{B})}$. Consequently, $(\|\ell\|, \|\ell'\|) \in (S^{-1})^{\mathcal{I}(\mathcal{B})}$. For (2b) suppose that $(\|\ell\|, \|\ell'\|) \in (S^{-1})^{\mathcal{I}(\mathcal{B})}$ and $\ell : \neg\exists S^{-1}.D \in \mathcal{B}$. As all the occurrences of the inverse operator have been pushed through other role connectives and double occurrences of $^{-1}$ have been removed¹ we can assume that $S = r$ for some role name r . Hence, $(\|\ell'\|, \|\ell\|) \in r^{\mathcal{I}(\mathcal{B})}$ and, by the definition of $r^{\mathcal{I}(\mathcal{B})}$, there is $\ell'' \sim_{\mathcal{B}} \ell$ such that $\ell' : \exists r.\{\ell''\} \in \mathcal{B}$. By the (mon) rule, $\ell'' : \neg\exists r^{-1}.D \in \mathcal{B}$. Finally, by the $(\neg\exists^{-1})$ rule, $\ell' : \neg D$ is in the branch \mathcal{B} .

$R = S_0 \sqcup S_1$. For (2a) suppose $\ell : \exists(S_0 \sqcup S_1).\{\ell'\} \in \mathcal{B}$. Hence, $\ell : \exists S_0.\{\ell'\} \in \mathcal{B}$ or $\ell : \exists S_1.\{\ell'\} \in \mathcal{B}$ by the $(\exists \sqcup)$ rule. Thus, by the induction hypothesis for $S_0 \prec R$ and $S_1 \prec R$ we have either $(\|\ell\|, \|\ell'\|) \in S_0^{\mathcal{I}(\mathcal{B})}$ or $(\|\ell\|, \|\ell'\|) \in S_1^{\mathcal{I}(\mathcal{B})}$. Finally, by the semantics of the relational \sqcup connective we obtain $(\|\ell\|, \|\ell'\|) \in (S_0 \sqcup S_1)^{\mathcal{I}(\mathcal{B})}$. For (2b) let $(\|\ell\|, \|\ell'\|) \in (S_0 \sqcup S_1)^{\mathcal{I}(\mathcal{B})} = S_0^{\mathcal{I}(\mathcal{B})} \cup S_1^{\mathcal{I}(\mathcal{B})}$ and $\ell : \neg\exists(S_0 \sqcup S_1).D \in \mathcal{B}$. By the $(\neg\exists \sqcup)$ rule we obtain that both $\ell : \neg\exists S_0.D$ and $\ell : \neg\exists S_1.D$ are in \mathcal{B} . Therefore, by the induction hypothesis for $S_0 \prec R$ and $S_1 \prec R$ the property (2b) holds and, thus, we have $\ell' : \neg D \in \mathcal{B}$.

$R = \neg S$. For (2a) suppose $\ell : \exists\neg S.\{\ell'\} \in \mathcal{B}$. Then $\ell : \neg\exists S.\{\ell'\} \in \mathcal{B}$ is obtained with the $(\exists\neg)$ rule. If $(\|\ell\|, \|\ell'\|) \notin (\neg S)^{\mathcal{I}(\mathcal{B})}$ then $(\|\ell\|, \|\ell'\|) \in S^{\mathcal{I}(\mathcal{B})}$ and by property (2b) which holds by the induction hypothesis for $S \prec R$ we have that $\ell' : \neg\{\ell'\}$ is in \mathcal{B} . This concept together with $\ell' : \{\ell'\}$ implies the branch is closed. We reach a contradiction, so $(\|\ell\|, \|\ell'\|) \in (\neg S)^{\mathcal{I}(\mathcal{B})}$.

For property (2b) suppose that $(\|\ell\|, \|\ell'\|) \in (\neg S)^{\mathcal{I}(\mathcal{B})}$ and $\ell : \neg\exists\neg S.D$ are in the branch \mathcal{B} . Then we have $(\|\ell\|, \|\ell'\|) \notin S^{\mathcal{I}(\mathcal{B})}$ and, hence, by the contrapositive of property (2a) for $S \prec R$, $\ell : \exists S.\{\ell'\}$ is not in \mathcal{B} . Applying the $(\neg\exists\neg)$ rule to $\ell : \neg\exists\neg S.D$ we get $\ell' : \neg D \in \mathcal{B}$.

□

A direct consequence of this lemma is a stronger form of completeness of the tableau calculus.

Theorem 7 (Completeness). *For any set of concepts \mathcal{S} and any tableau $T_{\mathcal{ALBO}^{\text{id}}}(\mathcal{S})$, if there is an open branch in $T_{\mathcal{ALBO}^{\text{id}}}(\mathcal{S})$ then \mathcal{S} is satisfiable in an $\mathcal{ALBO}^{\text{id}}$ -model.*

This implies $T_{\mathcal{ALBO}^{\text{id}}}$ is a complete tableau calculus for testing satisfiability of concept expressions in $\mathcal{ALBO}^{\text{id}}$.

Thus, the calculus $T_{\mathcal{ALBO}^{\text{id}}}$ is sound and complete for reasoning in $\mathcal{ALBO}^{\text{id}}$.

¹Removing double occurrences of the inverse operator in front of atoms is not essential for the proof but it simplifies matters a bit.

1. $\ell_0 : \Box \exists r.p$ given
2. $\ell_0 : \{\ell_0\}$ (refl),1
3. $\ell_0 : \exists r.p$ (\Box),1,2
4. $\ell_1 : p$ (\exists),3
5. $\ell_0 : \exists r.\{\ell_1\}$ (\exists),3
6. $\ell_1 : \{\ell_1\}$ (refl),4
7. $\ell_1 : \exists r.p$ (\Box),1,6
8. $\ell_0 \not\sim \ell_1$ Loop checking: $p \in \mathcal{L}(\ell_1) \setminus \mathcal{L}(\ell_0)$
9. $\ell_2 : p$ (\exists),7
10. $\ell_1 : \exists r.\{\ell_2\}$ (\exists),7
11. $\ell_2 : \{\ell_2\}$ (refl),9
12. $\ell_2 : \exists r.p$ (\Box),1,11
13. $\ell_1 \sim \ell_2$ Loop checking: $\mathcal{L}(\ell_1) \supseteq \mathcal{L}(\ell_2)$

Figure 3: Standard loop checking mechanism in \mathcal{ALCCO}^\Box .

6 Unrestricted Blocking

There are satisfiable concepts which can result in an infinite $T_{\mathcal{ALBO}}^{\text{id}}$ -tableau, where all open branches are infinite. The concept

$$\neg \exists (s \sqcup \neg s). \neg \exists r.p$$

is such an example. Since the prefix $\neg \exists (s \sqcup \neg s). \neg$ is equivalent to the universal modality, the concept $\ell : \exists r.p$ is propagated to every individual ℓ in every branch of the tableau. The concept $\ell : \exists r.p$ itself, each time triggers the creation of a new individual with the (\exists) rule. Thus, any branch of the tableau contains infinitely many individuals. The branches have however a regular structure that can be detected with loop detection or blocking mechanisms.

Let us demonstrate how *standard loop checking* (in this case, subset ancestor blocking) detects a loop in this example. As we already observed, satisfiability of the concept $\neg \exists (s \sqcup \neg s). \neg \exists r.p$ corresponds to satisfiability of the concept $\Box \exists r.p$ in the description logic \mathcal{ALCCO}^\Box which is \mathcal{ALCO} with the universal modality \Box . As a calculus for \mathcal{ALCCO}^\Box we use the rules for \mathcal{ALCO} from Figure 1 plus the following two rules (where ℓ' in the left rule is uniquely associated with $\ell : \neg \Box C$).

$$(\neg \Box): \frac{\ell : \neg \Box C}{\ell' : \neg C} \quad (\ell' \text{ is new}) \qquad (\Box): \frac{\ell : \Box C, \ell' : \{\ell'\}}{\ell' : C}$$

This calculus is sound and complete.

Figure 3 gives a derivation in this calculus using loop checking based on standard subset ancestor blocking. For any individual ℓ , let $\mathcal{L}(\ell)$ be a set of concepts associated with ℓ in the current branch.

It is defined by

$$\mathcal{L}(\ell) \stackrel{\text{def}}{=} \{C \mid \ell : C \text{ is in the current branch, and } C \neq \{\ell\} \text{ and } C \neq \exists r.\{\ell'\}\}.$$

After all the type-completing rules have been applied to all concept expressions labelled with a specific individual, loop checking tests are performed relative to an ancestor individual. Two loop checking tests are performed, namely in step 8 and step 13. Consider step 13. All the type-completing rules have been applied to all concept expressions of the form $\ell_2 : C$ and $\ell_1 : C$. Comparison of the sets of concepts $\mathcal{L}(\ell_1)$ and $\mathcal{L}(\ell_2)$ associated with ℓ_1 and ℓ_2 in the loop checking test shows that $\mathcal{L}(\ell_1)$ subsumes $\mathcal{L}(\ell_2)$. Thus they are in a subset relationship as indicated, and consequently the individuals ℓ_1 and ℓ_2 can be identified. At step 8 the sets $\mathcal{L}(\ell_0)$ and $\mathcal{L}(\ell_1)$ are not in a subset relationship because $\ell_0 : p$ is not present in the branch. The derivation therefore cannot yet stop, but does in step 13.

This example illustrates one of the simplest forms of standard loop checking used in description and modal logic tableau procedures. Other forms of loop-checking have been devised for different logics. One can classify the different existing loop-checking mechanisms as using a combination of these blocking techniques: subset or equality blocking, non-pairwise or pairwise blocking, ancestor or anywhere blocking, and static or dynamic blocking. These techniques are all based on comparing sets of concepts labelled by some individuals (and in the case of pairwise blocking also sets of roles associated to predecessors). None of these forms of loop checking are sufficient to handle role negation though.

Suppose that, at some node of a branch, all the type-completing rules have been applied to concepts labelled with individuals ℓ_0 and ℓ_1 . Following the standard non-pairwise blocking procedure we can identify ℓ_0 and ℓ_1 if they both label the same set of concepts. However, it is not correct to start identifying the individuals at this point in the derivation as there could be, for example, a concept $\exists S.\neg\exists\neg R.C$ involving role negation, where R and C are expressions that are suitably complex. At a subsequent point the (\exists) rule is applied to this concept and, hence, the expression $\ell_2 : \neg\exists\neg R.C$, where ℓ_2 is new, appears in the branch. This triggers the application of the $(\neg\exists\neg)$ rule and because of the form of R and $\neg C$ in the two branches it also triggers application of the $(\neg\exists)$ rule and possibly other type-completing rules. After such applications, it can happen that the labels are not identifiable (anymore) because their types have become distinguishable by some concept. That is, any introduction of a new individual in a tableau has, in general, a global effect on the provisional model constructed so far.

The example in Figure 4 illustrates this global effect. Here we are interested in the satisfiability of

$$(\exists r.p) \sqcap (\exists s.p) \sqcap (\exists t.\forall t.\forall s^{-1}.\perp) \sqcap (\forall t.\bar{\forall}t.\forall s^{-1}.\perp).$$

At step 10 none of the type-completing rules need to be applied to concepts labelled with ℓ_1 and ℓ_2 . Although at this point ℓ_1 and ℓ_2 are labels of the same subconcepts of the given concepts, we cannot make them equal (using equality anywhere blocking). The reason is that in step 11 a new individual is introduced which causes a few applications of the $(\neg\exists)$ rule, and as a result, at step 21, the types of ℓ_1 and ℓ_2 are now distinguished by the concept $\exists s^{-1}.(p \sqcup \neg p)$. The example is designed to show that it is not possible to use standard blocking techniques to identify individuals (ℓ_1 and ℓ_2) although they are locally complete (that is, none of the type completing rules can be applied to them), because other individuals (ℓ_0) can influence ℓ_1 and ℓ_2 via the right branch of the $(\neg\exists\neg)$ rule.

The examples illustrate that a reason for non-termination of T_{ALBO}^{id} is the possible infinite generation of labels. Although the rules respect the well-founded ordering \prec they do it only with

respect to expressions, not labelled expressions (assertions). Furthermore, it is easy to see that only applications of the (\exists) rule generate new individuals in the branch. Thus, a reason that a branch can be infinite is the unlimited application of the (\exists) rule; it is in fact the only possible reason. It is crucial therefore to have a blocking mechanism that avoids infinite derivations by restricting the application of the (\exists) rule.

In order to turn the calculus $T_{\mathcal{ALBO}^{\text{id}}}$ into a terminating calculus for $\mathcal{ALBO}^{\text{id}}$, we introduce a new approach to blocking.

Let $<$ be an ordering on individuals in the branch which is a linear extension of the order of introduction of the individuals during the derivation. That is, let $\ell < \ell'$, whenever the first appearance of individual ℓ' in the branch is strictly later than the first appearance of individual ℓ .

We add the following rule, called the *unrestricted blocking* rule, to the calculus.

$$(\text{ub}): \frac{\ell : \{\ell\}, \ell' : \{\ell'\}}{\ell : \{\ell'\} \mid \ell : \neg\{\ell'\}}$$

Moreover, we require that the following conditions both hold.

- (c1) If $\ell : \{\ell'\}$ appears in a branch and $\ell < \ell'$ then possible applications of the (\exists) rule to expressions of the form $\ell' : \exists R.C$ are not performed within the branch.
- (c2) In every open branch there is some node from which point onwards before any application of the (\exists) rule all possible applications of the (ub) rule have been performed.

The intuition of the (ub) rule is that it conjectures whether two labels are equal or not. In the left branch two labels are set to be equal. If this does not lead to finding a model then the labels cannot be equal. This is the information carried by the right branch. Conditions (c1) and (c2) are important to restrict the application of the (\exists) rule. Condition (c1) specifies that it may only be applied to labelled expressions where the label is the smallest representative of an equivalence class. Condition (c2) says that at some point in a branch blocking has been applied exhaustively before the application of the (\exists) rule.

We use the notation $T_{\mathcal{ALBO}^{\text{id}}}(\text{ub})$ for the extension of $T_{\mathcal{ALBO}^{\text{id}}}$ with this blocking mechanism using the (ub) rule.

Theorem 8. $T_{\mathcal{ALBO}^{\text{id}}}(\text{ub})$ is a sound and complete tableau calculus for $\mathcal{ALBO}^{\text{id}}$.

Proof. The (ub) rule is sound in the usual sense. The blocking conditions (c1) and (c2) are sound in the sense that they cannot cause an open branch to become closed. Hence, the blocking rule and the blocking conditions can be safely added to *any* tableau calculus without endangering soundness or completeness. As $T_{\mathcal{ALBO}^{\text{id}}}$ is sound and complete, it follows that $T_{\mathcal{ALBO}^{\text{id}}}(\text{ub})$ is sound and complete. \square

7 Termination Through Unrestricted Blocking

Next we prove termination. Since every finite set of concepts can be replaced by the conjunction of its elements, without loss of generality in this section and the next section, we restrict ourselves to a single input concept.

Let C be the given input concept and n be the length of the concept C . Let k be the number of individuals occurring in the input concept assertion $\ell : C$. Suppose \mathcal{B} is an arbitrary open branch in a $T_{\mathcal{ALBO}^{\text{id}}}(\text{ub})$ tableau for C and $\mathcal{I}(\mathcal{B})$ be the model constructed from \mathcal{B} as described in Section 5.

For every $\|\ell\| \in \Delta^{\mathcal{I}(\mathcal{B})}$, let $\#\exists(\|\ell\|)$ denote the number of applications in \mathcal{B} of the (\exists) rule to concepts of the form $\ell' : \exists R.D$ with $\ell' \in \|\ell\|$.

Lemma 9. $\#\exists(\|\ell\|)$ is finite for every $\|\ell\| \in \Delta^{\mathcal{I}(\mathcal{B})}$.

Proof. Suppose not, that is, $\#\exists(\|\ell\|)$ is infinite. The number of concepts of the form $\exists R.D$ where D is not a singleton under labels in the branch is finitely bounded. Because the (\exists) rule is not allowed to be applied to role assertions (that is, concepts of the form $\ell : \exists R.\{\ell'\}$) there is a sequence of individuals ℓ_0, ℓ_1, \dots such that every $\ell_i \in \|\ell\|$ and the (\exists) rule has been applied to concepts $\ell_0 : \exists R.D, \ell_1 : \exists R.D, \dots$ for some $\exists R.D \preceq C$. However, such a situation is impossible because of conditions (c2) and (c1). Indeed, without loss of generality we can assume that $\ell < \ell_0 < \ell_1 < \dots$. Then, by Condition (c2), starting from some node of \mathcal{B} , as soon as ℓ_i appears in \mathcal{B} , it is detected that $\ell_i \in \|\ell\|$ before any next application of the (\exists) rule and, hence, ℓ_i is immediately blocked for any application of the rule (\exists) , due to Condition (c1). \square

Let $\#\exists(\mathcal{B})$ denote the number of applications of the (\exists) rule in a branch \mathcal{B} . Since the (\exists) rule is the only rule that generates new individuals, $\#\exists(\mathcal{B})$ is in fact the number of generated individuals in the branch \mathcal{B} and the number of all individuals occurring in \mathcal{B} is $k + \#\exists(\mathcal{B})$.

Lemma 10. If $\text{Card}(\Delta^{\mathcal{I}(\mathcal{B})})$ is finite then $\#\exists(\mathcal{B})$ is finite and does not exceed

$$\max\{\#\exists(\|\ell\|) \mid \|\ell\| \in \Delta^{\mathcal{I}(\mathcal{B})}\} \cdot \text{Card}(\Delta^{\mathcal{I}(\mathcal{B})}).$$

Proof. Since $\Delta^{\mathcal{I}(\mathcal{B})}$ is finite and by Lemma 9 the value $\max\{\#\exists(\|\ell\|) \mid \|\ell\| \in \Delta^{\mathcal{I}(\mathcal{B})}\}$ exists and is finite. Thus, we have $\#\exists(\mathcal{B}) \leq \max\{\#\exists(\|\ell\|) \mid \|\ell\| \in \Delta^{\mathcal{I}(\mathcal{B})}\} \cdot \text{Card}(\Delta^{\mathcal{I}(\mathcal{B})})$. \square

Furthermore, the following lemma holds.

Lemma 11. The number of concepts in \mathcal{B} does not exceed

$$n \cdot (k + \#\exists(\mathcal{B})) + 2n \cdot (k + \#\exists(\mathcal{B}))^2.$$

In particular, if $\#\exists(\mathcal{B})$ is finite then \mathcal{B} is finite.

Proof. Let $D \in \mathcal{B}$. Then D has one of the following forms:

$$\begin{aligned} D &= \ell : D', \quad \text{where } D' \text{ is a concept in } \text{sub}(C), \\ D &= \ell : \{\ell'\}, \\ D &= \ell : \neg\{\ell'\}, \\ D &= \ell : \exists R.\{\ell'\}, \quad \text{where } R \text{ is a role in } \text{sub}(C), \\ D &= \ell : \neg\exists R.\{\ell'\}, \quad \text{where } R \text{ is a role in } \text{sub}(C). \end{aligned}$$

The number of individuals and concepts in $\text{sub}(C)$ is bounded by n and the number of roles in $\text{sub}(C)$ is strictly less than n . Therefore, an upper bound for the number of concepts in the branch \mathcal{B} can be given as: $n \cdot (k + \#\exists(\mathcal{B})) + (k + \#\exists(\mathcal{B}))^2 + (k + \#\exists(\mathcal{B}))^2 + (n - 1) \cdot (k + \#\exists(\mathcal{B}))^2 + (n - 1) \cdot (k + \#\exists(\mathcal{B}))^2 = n \cdot (k + \#\exists(\mathcal{B})) + 2n \cdot (k + \#\exists(\mathcal{B}))^2$. \square

As a consequence of Lemmas 10 and 11 we obtain the following statement.

Corollary 12. *Let \mathcal{B} be an open branch in a $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}$ tableau for a concept C . Then the branch \mathcal{B} is finite iff $\Delta^{\mathcal{I}(\mathcal{B})}$ is finite.*

Lemma 13. *Assume that the input concept C is satisfiable in a model \mathcal{I} . Then there is an open branch \mathcal{B} in $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}(C)$ such that $\text{Card}(\Delta^{\mathcal{I}(\mathcal{B})}) \leq \text{Card}(\Delta^{\mathcal{I}})$.*

Proof. By induction on application of the rules in $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}(C)$ we choose an appropriate branch \mathcal{B} and construct a model \mathcal{J} which differs from \mathcal{I} only in the interpretations of individuals occurring in \mathcal{B} but not occurring in C , and so that $\mathcal{J} \models \ell : D$ for every concept $\ell : D$ from \mathcal{B} .

More precisely, let $\Delta^{\mathcal{J}} \stackrel{\text{def}}{=} \Delta^{\mathcal{I}}$ and $p^{\mathcal{J}} \stackrel{\text{def}}{=} p^{\mathcal{I}}$ and $r^{\mathcal{J}} \stackrel{\text{def}}{=} r^{\mathcal{I}}$ for every concept symbol p and role symbol r . Additionally $\ell^{\mathcal{J}} \stackrel{\text{def}}{=} \ell^{\mathcal{I}}$ for every individual ℓ which either occurs in C or does not occur in \mathcal{B} . By induction on the application of the rules in $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}(C)$ we expand the interpretation \mathcal{J} to all individuals occurring in \mathcal{B} such that $\mathcal{J} \models \ell : D$ for every concept $\ell : D$ from \mathcal{B} .

- For the base case of the induction argument we choose the initial node \mathcal{S}_0 of the branch \mathcal{B} as follows: $\mathcal{S}_0 \stackrel{\text{def}}{=} \{\ell_0 : C\}$ where ℓ_0 is a new individual. Since C is satisfiable in \mathcal{I} , there is w in $C^{\mathcal{I}}$. We let $\ell_0^{\mathcal{J}} \stackrel{\text{def}}{=} w$.
- The induction step involves considering cases for each of the rules in the $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}$ calculus. We consider only two rules, namely (\exists) and (ub) rules.
 - (\exists) Suppose $\ell : \exists R.D$ is in the node \mathcal{S}_n and $\ell : \exists R.\{\ell'\}$ and $\ell' : D$ are the conclusions of the rule. We set $\mathcal{S}_{n+1} \stackrel{\text{def}}{=} \mathcal{S}_n \cup \{\ell : \exists R.\{\ell'\}, \ell' : D\}$. By the induction hypothesis we have that $\mathcal{J} \models \ell : \exists R.D$. That is, there is an element w in $\Delta^{\mathcal{J}}$ such that $(\ell^{\mathcal{J}}, w) \in R^{\mathcal{J}}$ and $w \in D^{\mathcal{J}}$. We set $\ell'^{\mathcal{J}} \stackrel{\text{def}}{=} w$.
 - (ub) Suppose $\ell : \{\ell\}$ and $\ell' : \{\ell'\}$ are in the node \mathcal{S}_n . By the induction hypothesis $\ell^{\mathcal{J}}$ and $\ell'^{\mathcal{J}}$ are both defined. We have two cases: either $\ell^{\mathcal{J}} = \ell'^{\mathcal{J}}$ or $\ell^{\mathcal{J}} \neq \ell'^{\mathcal{J}}$. We let $\mathcal{S}_{n+1} \stackrel{\text{def}}{=} \mathcal{S}_n \cup \{\ell : \{\ell'\}\}$ in the first case and $\mathcal{S}_{n+1} \stackrel{\text{def}}{=} \mathcal{S}_n \cup \{\ell : \neg\{\ell'\}\}$ in the second case.

Thus, we constructed the branch \mathcal{B} and a model \mathcal{J} with the required properties. \mathcal{B} is open since the set of all concepts from \mathcal{B} is satisfiable in the model \mathcal{J} . Now let f be a function mapping $\Delta^{\mathcal{J}}$ ($= \Delta^{\mathcal{I}}$) to $\Delta^{\mathcal{I}(\mathcal{B})}$ and be defined by:

$$f(w) \stackrel{\text{def}}{=} \begin{cases} \|\ell\|, & \text{if } w = \ell^{\mathcal{J}} \text{ for some individual } \ell \text{ in } \mathcal{B}, \\ \text{arbitrary in } \Delta^{\mathcal{I}(\mathcal{B})}, & \text{otherwise.} \end{cases}$$

By construction of \mathcal{J} if $w = \ell^{\mathcal{J}} = \ell'^{\mathcal{J}}$ for two different individuals ℓ and ℓ' from the branch \mathcal{B} then $\ell : \{\ell'\} \in \mathcal{B}$ and, hence, $\|\ell\| = \|\ell'\|$. The function f is thus defined correctly. Furthermore, f is onto $\Delta^{\mathcal{I}(\mathcal{B})}$ because $\Delta^{\mathcal{I}(\mathcal{B})} = \{\|\ell\| \mid \ell : \{\ell'\} \in \mathcal{B}\}$. \square

Combining this lemma with Theorem 3 we obtain the following theorem.

Theorem 14. *Let C be a satisfiable concept and n be the length of C . Then in every $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}$ tableau for C , there exists an open branch \mathcal{B} such that $\text{Card}(\Delta^{\mathcal{I}(\mathcal{B})}) \leq \mu(n)$.*

Theorem 15 (Termination). *$T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}$ is a terminating tableau calculus for satisfiability in $\mathcal{ALBO}^{\text{id}}$.*

Proof. In any $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}$ -tableau, every closed branch is finite. Suppose there is an open branch in the tableau. By the Completeness Theorem 7 this means that the input concept is satisfiable. By Theorem 14 and Corollary 12 there is a finite open branch in the tableau. \square

Corollary 16. *Any non-deterministic tableau procedure based on $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}$ is a decision procedure for $\mathcal{ALBO}^{\text{id}}$ and all its sublogics.*

Notice that Condition (c2) is essential for ensuring termination of a $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}$ derivation. Figure 5 shows that without (c2) the $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}$ tableau for the concept

$$\neg\exists(s \sqcup \neg s). \neg\exists r.p$$

would not terminate because new individuals are generated more often than the equality conjectures made via the (ub) rule.

In addition, omitting Condition (c1) causes any tableau for the concept $\ell : \exists r.p$ not to terminate. Thus, the conditions (c1) and (c2) are both necessary conditions for termination of the calculus $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}$.

8 A Complexity-Optimal Tableau Decision Procedure

Our estimations, in this section, for the complexity of tableau procedures based on the $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}$ calculus rely on the following two observations. First, from Theorem 14 it follows that, in a practical decision procedure based on $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}$, we can ignore any branch \mathcal{B} where the cardinality of $\Delta^{\mathcal{I}(\mathcal{B})}$ is larger than $\mu(n)$. Any such branch is either closed or there is an open branch with a smaller number of classes of equal individuals.

Second, Lemmas 11 and 10 provide a clue for estimating the number of derivation steps (that is, the number of rule applications) in a shortest open branch \mathcal{B} . Since the maximal number of premises of a rule is two, and none of the rules are applied to the same set of premises more than once, if N is a number of concepts in \mathcal{B} , then the number of derivation steps in \mathcal{B} is less than or equal to N^2 . Thus, the following lemma holds.

Lemma 17. *Let \mathcal{B} be a finite open branch in $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}(C)$ and*

$$M = \max\{\#\exists(\|\ell\|) \mid \|\ell\| \in \Delta^{\mathcal{I}(\mathcal{B})}\}.$$

Then the number of derivation steps in \mathcal{B} does not exceed

$$(*) \quad \left(n \cdot (k + M \cdot \mu(n)) + 2n \cdot (k + M \cdot \mu(n))^2 \right)^2.$$

In order to define a complexity-optimal tableau decision procedure, let us define the following derivation strategy.

Avoid huge branch strategy: Limit the number of derivation steps in every branch of a tableau derivation to the bound (*) in Lemma 17 and do not expand branches that become longer.

Whereas n and k depend only on the given concept C , the value $M = \max\{\#\exists(\|\ell\|) \mid \|\ell\| \in \Delta^{\mathcal{I}(\mathcal{B})}\}$ depends also on the particular strategy of applying the (ub) rule. In fact, the following lemma holds.

Lemma 18. *In a branch \mathcal{B} let M' be the number of individuals occurring strictly before the node from which point onwards, before any application of the (\exists) rule, all applications of the (ub) rule have been performed in accordance with Condition (c2). Let n' be the number of concepts of the form $\exists R.D$ in $\text{sub}(C)$. Then $M \leq M' + n'$.*

Proof sketch. It can be seen from the proof of Lemma 9 that, after the node where exhaustive applications of the (ub) rule started, the number of applications of the (\exists) rule to concepts labelled by individuals from the same equivalence class in the branch \mathcal{B} is restricted by n' . The number of applications of the (\exists) rule to concepts before this node is restricted by the number of individuals which are present before that node, that is, by M' . \square

We note that $n' \leq n$, but, depending on the strategy used, M' can have any value. However, if a tableau strategy ensures that M' is at most exponential in the length of the given concept C then the number of derivation steps in the branch \mathcal{B} is also at most exponential in the length of C . This is in particular true if Condition (c2) is satisfied for the *first* node of the tableau. In this case, $M' = 0$. This proves the following theorem.

Theorem 19. *Any non-deterministic tableau procedure based on $T_{\mathcal{ALBO}^{\text{id}}}(\text{ub})$ runs in NExpTime, when it uses the ‘avoid huge branch strategy’ together with the tableau expansion strategy ensuring that before any application of the (\exists) rule all possible applications of the (ub) rule have been performed.*

This gives us a complexity optimal tableau decision procedure for $\mathcal{ALBO}^{\text{id}}$.

9 Deterministic Decision Procedures

The presented tableau calculus $T_{\mathcal{ALBO}^{\text{id}}}(\text{ub})$ provides the basis for non-deterministic decision procedures.

When turning $T_{\mathcal{ALBO}^{\text{id}}}(\text{ub})$ into a deterministic decision procedure it is crucial that this is done in a fair way. A procedure is *fair* if, when an inference is possible forever, then it is performed eventually. This means a deterministic tableau procedure based on $T_{\mathcal{ALBO}^{\text{id}}}(\text{ub})$ may not defer the use of an applicable rule indefinitely. In our context fairness must be understood in a *global* sense. That is, a tableau procedure has to be fair not only to expressions in a particular branch but to expressions in *all* branches of a tableau. In other words, a procedure is fair if it makes both the branch selection, and the selection of expressions to which to apply a rule, in a fair way.

Theorem 20. *Any fair tableau procedure based on $T_{\mathcal{ALBO}^{\text{id}}}(\text{ub})$ is a decision procedure for $\mathcal{ALBO}^{\text{id}}$ and all its sublogics.*

To illustrate the importance of fairness we give two examples. The concept

$$\neg\left(\exists(s \sqcup \neg s). \neg\exists r.p \sqcup \neg\exists t. \neg\exists r.p\right),$$

or equivalently $(\Box\exists r.p) \sqcap (\exists t.\forall r.\neg p)$, is not satisfiable. Figure 6 gives a depth-first left-to-right derivation that is unfair and does not terminate. It can be seen that the derivation is infinite because the application of the (\exists) rule to $\ell_0 : \exists t.\neg\exists r.p$ is deferred forever and, consequently, a contradiction is not found. This illustrates the importance of fairness for (refutational) completeness.

The next example illustrates the importance of fairness for termination. The concept

$$\neg\exists(s \sqcup \neg s).\neg\exists r.p,$$

or equivalently $\Box\exists r.p$, is satisfiable. The derivation in Figure 7 is obtained with a depth-first *right-to-left* strategy. However, the repeated selection of the right branch at (ub) choice points, in particular, causes all the individuals in the branch to be pair-wise non-equal. The concept $\ell : \exists r.p$ re-appears repeatedly for every individual ℓ in the branch. This triggers the repeated generation of a new individual by the (\exists) rule, resulting in an infinite derivation. This strategy is unfair because all branches except for the right-most branch get ignored. In the branches right-most with respect to the (ub) rule, it is as if blocking with the (ub) rule has never been applied.

Recall from Section 4, a tableau calculus is terminating iff the tableau constructed for any finite concept set \mathcal{S} is finite whenever $T(\mathcal{S})$ is closed or contains a finite open branch if the tableau is open. For a satisfiable concept set this means it is possible to construct a finite, fully expanded, open branch. Not all branches in an open terminating tableau need to be finite though. Reker [2011] gives small examples expressible in $\mathcal{ALBO}^{\text{id}}$ for which the left-most open branch in a tableau derivation is infinite.

This means an immediate way of obtaining a fair deterministic decision procedure for a terminating tableau is to use breadth-first search. Another possibility is to use depth-first search with iterative deepening. The idea of iterative deepening search is that the tableau is expanded up to a fixed depth. If a closed tableau is found or if an open fully expanded branch is found then the derivation process stops. If not, then the depth is increased and the tableau is constructed up to this depth, and so on.

A benefit of breadth-first and depth-first iterative deepening strategies is the generation of small models. Because the entire tableau needs to be stored in memory the implementation is however more involved and more memory intensive, and potentially less efficient. Importantly though both breadth-first search and depth-first iterative deepening strategies provide the basis for the implementation of sound and complete tableau-based decision procedures for $\mathcal{ALBO}^{\text{id}}$.

Additionally, depth-first search combined with the ‘avoid huge branch strategy’ provides a fair, deterministic decision procedure.

Theorem 21. *The following are deterministic decision procedures for $\mathcal{ALBO}^{\text{id}}$ and all its sublogics:*

1. *Any fair tableau procedure based on $T_{\mathcal{ALBO}^{\text{id}}}(\text{ub})$ using a breadth-first search strategy.*
2. *Any fair tableau procedure based on $T_{\mathcal{ALBO}^{\text{id}}}(\text{ub})$ using a depth-first search strategy with iterative deepening.*
3. *Any fair tableau procedure based on $T_{\mathcal{ALBO}^{\text{id}}}(\text{ub})$ using a depth-first left-to-right strategy and the ‘avoid huge branch strategy’.*

Whether a depth-first left-to-right strategy without exploiting the model bound gives a decision procedure for $\mathcal{ALBO}^{\text{id}}$ is open. It is in particular open whether there is a way to force the left-most open branch to be finitely bounded.

10 Discussion

An important contribution of this paper is the unrestricted blocking mechanism based on the unrestricted blocking rule. It is conceptually similar but more generic than the approach of reusing

domain terms in, for example, Bry and Manthey [1988] and Bry and Torge [1998], where it is used to compute domain minimal models. The following rule

$$\frac{\ell : \exists R.C}{\ell : \exists R.\{\ell_0\}, \ell_0 : C \mid \dots \mid \ell : \exists R.\{\ell_n\}, \ell_n : C \mid \ell : \exists R.\{\ell'\}, \ell' : C'}$$

where ℓ_0, \dots, ℓ_n are all the individuals occurring in the current branch and ℓ' is a fresh individual, is an adaptation of the δ^* -rule from Bry et al. to the language of description logics. It is not difficult to see that every tableau derivation with this rule can be rewritten to an equivalent tableau derivation using the unrestricted blocking rule but not the other way around.

One of the advantages of the unrestricted blocking mechanism is that it allows to separate proofs of termination of tableau calculi from proofs of their soundness and completeness. In the description, modal and hybrid logic literature blocking mechanisms are usually tightly integrated with the tableau rules and the tableau procedure involving the use of status variables and specialised blocking tests. Soundness and completeness of the procedure is proved taking into account the given blocking mechanism. This creates intricacies and conceptual dependencies in the proofs. In contrast, blocking based on the (ub) rule is conceptually and intuitively simple. It is not tied to a specific logic. It can be added to any sound and complete tableau calculus while preserving soundness and completeness.

Exploiting this we show in Schmidt and Tishkovsky [2008] that any tableau procedure based on a tableau calculus extended with this blocking mechanism is guaranteed to terminate if some additional general conditions are satisfied. These are the blocking conditions (c1) and (c2) and fairness of a particular tableau strategy used in the tableau procedure, which is essential for the method to work. In addition, the effective finite model property must hold and it must be proved by a filtration argument correlating in an appropriate way with the original tableau calculus. The present paper provides a different kind of proof for termination to Schmidt and Tishkovsky [2008]. The proof can be extended to show termination of tableau calculi equipped with the unrestricted blocking mechanism for logics for which the effective model property may not hold but the *finite model property* still holds. Proving the finite model property can be involved but proofs of it may already be known. This means that known finite model property results can be readily exploited to define terminating tableau calculi. In Schmidt [2009] and Schmidt and Tishkovsky [2011] we have introduced methods for systematically developing semantic tableau calculi for modal and related logics. Adding unrestricted blocking turns many of the tableau calculi generated using these methods into terminating ones [Schmidt and Tishkovsky, 2011].

The way the blocking mechanism is defined in our calculus means that from some point onwards it needs to be used eagerly, as is required by Condition (c2). This restriction still leaves a lot of room for different tableau expansion strategies. For example, a tableau procedure can postpone applications of the (ub) rule or apply the rule in a non-eager way while the provisional model constructed in a branch is not too large. Subtle measures and strategies can be devised (for instance, based on run time or available memory) for determining when to start applying the blocking rule eagerly.

Eager application of the blocking rule produces small models. This is an advantage because small counter-examples can be produced whereas models generated by standard blocking mechanisms can be large. Smaller models consume less memory, meaning that memory consumption can be significantly smaller leading to quicker answers and improved performance.

On the other hand the blocking rule involves branching and excessive branching can degrade performance. It can thus be advantageous to apply blocking only selectively. Existing standard

blocking mechanisms only identify individuals if certain blocking conditions are true. In particular, they are based on inclusion or equality tests on sets of expressions that must normally be constructed in a certain way. Individuals are identified only implicitly through the use of status variables which identify individuals as blocked and blockable. The expansion is usually assumed to be depth-first and the rules are usually assumed to be applied in a certain order. The unrestricted blocking mechanism is less restrictive and is monotone: there is no blocking test, any individual is blockable and blocked individuals remain blocked in a branch. There are side-conditions for the application of the unrestricted blocking rule but they are designed to be as weak and permissive as possible. As is shown none of the conditions can be omitted.

Though standard blocking mechanisms are defined significantly differently, they can be viewed as specialisations and adaptations of unrestricted blocking. Standard blocking amounts to the restricted application of blocking, sometimes without change of the status of blocked individuals (static blocking), sometimes with change in the status of blocked individuals (dynamic blocking). Restricting applications of blocking reduces the number of branching nodes and the overall search space can be significantly smaller. A particular blocking test may need to be realised by an algorithm complicated enough to create considerable overhead. Ultimately, the trade-off between the resources required for blocking tests and excessive branching created as a result of applications of the blocking rule is inescapable and needs to be carefully considered.

Different standard blocking mechanisms are needed for different logics. While one blocking mechanism might work for a particular logic it can be incomplete for a more expressive logic. The advantage of unrestricted blocking is automatic soundness and completeness for any logic. Analysis of the termination proof in Schmidt and Tishkovsky [2008] however suggests that, in the case of a particular logic, an appropriate blocking test can be extracted from the definition of a filtrated model used in the proof of the effective finite model property for the logic. This opens a promising direction of further research for devising new blocking tests and blocking mechanisms in a systematic way.

Compared to the tableau calculi of the conference paper [Schmidt and Tishkovsky, 2007] there are a few differences in the calculus $T_{\mathcal{ALBO}^{\text{id}}}$. Besides the presence of rules for the identity role in $T_{\mathcal{ALBO}^{\text{id}}}$, the essential difference is that the calculi in Schmidt and Tishkovsky [2007] include this additional congruence rule:

$$\text{(bridge): } \frac{\ell : \exists R.\{\ell'\}, \ell' : \{\ell''\}}{\ell : \exists R.\{\ell''\}}.$$

The results of the present paper show that this rule is superfluous for completeness of $T_{\mathcal{ALBO}^{\text{id}}}$. However the (bridge) rule is sound for $\mathcal{ALBO}^{\text{id}}$. It is not difficult to prove that the (bridge) rule is not derivable in $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}$. Therefore, the (bridge) rule is admissible but not derivable in $T_{\mathcal{ALBO}^{\text{id}}}$ and $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}$.

Our use of the equality constraints $\ell : \{\ell\}$ in the tableau rules is non-standard and employs an idea proposed and implemented in METTEL tableau prover [Tishkovsky, 2005, Hustadt et al., 2006, Tishkovsky et al., 2011]. Normally these labelled concepts would be deleted by standard simplification rules since $\ell : \{\ell\}$ is a validity. This would not be correct in our calculus because we use these equality constraints as *domain predicates* for keeping track of the individuals *essential* for instantiation by the γ rules in the calculus, and ultimately, for keeping track of the individuals essential for the construction of the model in a tableau branch. Here, by γ rules we mean the rules for universally quantified expressions creating, by repeated application, all instantiations of individuals occurring in conclusions but not in the premise (that is, the implicit universally quantified first-order variables). The γ rules of the $T_{\mathcal{ALBO}^{\text{id}}(\text{ub})}$ calculus are the $(\neg\exists\neg)$ rule and the (ub) rule.

Naively these rules can be defined to use all individuals occurring in the input set and all individuals introduced by the (\exists) rule for instantiation but this would create unnecessarily many instances and would mean the search space becomes unnecessarily large. The $(\neg\exists)$ rule does not require domain predication because this is already achieved by the second premise of the rule. Domain predication can be obtained in other ways by using an explicit concept name representing the domain as is done by Hustadt and Schmidt [1999], Schmidt [2006a], Baumgartner and Schmidt [2006]. If preferred, domain predication can be omitted and captured by appropriate side-conditions, but then the (refl) rule and the γ rules need to be reformulated and the proofs adapted slightly.

Although the proofs in this paper are relatively concise, they are not trivial. They are based on the idea that the finite model property for $\mathcal{ALBO}^{\text{id}}$ and termination property of the proposed tableau calculus are of the same nature, which is an important contribution. In particular, it was proved that a finite model (if it exists) can be generated by applying the unrestricted blocking rule eagerly within the branch.

While our tableau approach is in line with existing semantic tableau approaches described in the description, modal and hybrid logic literature, our presentation departs in significant ways from other presentations. To make the approach and the results as general as possible, we have presented it in terms of an abstract deduction calculus given by a set of inference rules without making any assumptions about how branch selection is performed, how tableau derivations are constructed, in which order the rules are applied, which individuals are blockable and the need for using status variables. Another difference is that we view a tableau to be a (tree) derivation rather than a model as is the tendency in the description logic literature. In our context models are in general not tree models or even tree-like, because $\mathcal{ALBO}^{\text{id}}$ does not have the tree-model property and the unrestricted blocking rule can identify any pair of individuals meaning that any structural properties, like being a tree model, are easily lost. This shift in perspective is essential for developing tableau approaches for description logics without a form of tree model property.

Practical considerations such as branch selection order, rule application order, search strategies and heuristics are important considerations when embarking on an implementation. We touched on practical aspects and described general notions of fairness providing minimal conditions to guarantee sound, complete and terminating tableau procedures. The standard optimisations such as simplification, backjumping, dynamic backtracking, different heuristics for branch selection and rule selection, lemma generation, et cetera, are all compatible with the presented calculi and procedures. An obvious simplification, for example, is the on-the-fly removal of double negations from concepts, and especially from roles, as this reduces a number of applications of the $(\neg\exists\neg)$ rule.

The unrestricted blocking mechanism has been implemented in METTEL tableau prover [Tishkovsky, 2005, Hustadt et al., 2006, Tishkovsky et al., 2011] and the MSPASS first-order resolution theorem prover. Tests with various description logic problems and problems in the TPTP library show that the general termination mechanism by means of the unrestricted blocking rule does work in practice [Baumgartner and Schmidt, 2008]. A more thorough discussion of these implementations and other practical aspects, let alone performance results, is beyond the scope of the paper.

11 Conclusion

We have presented a new, general tableau approach for deciding expressive description logics with complex role operators, including ‘non-safe’ occurrences of role negation. The tableau decision procedures found in the description logic literature, and implemented in existing tableau-based description logic systems, can handle a large class of description logics, but cannot currently handle

description logics with full role negation such as \mathcal{ALB} , \mathcal{ALBO} or $\mathcal{ALBO}^{\text{id}}$. The present paper closes this gap. The introduced blocking mechanism opens the way for improving the OWL 2 standard to cover larger decidable fragments of first-order logic and larger fragments of natural language than it currently does, with obvious benefits for applications of the semantic web.

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1. $\ell_0 : \exists r.p$ given
2. $\ell_0 : \exists s.p$ given
3. $\ell_0 : \exists t.\neg\exists t.\exists s^{-1}.(p \sqcup \neg p)$ given
4. $\ell_0 : \neg\exists t.\exists\neg t.\neg\exists s^{-1}.(p \sqcup \neg p)$ given
5. $\ell_1 : p$ $(\exists),1$
6. $\ell_0 : \exists r.\{\ell_1\}$ $(\exists),1$
7. $\ell_1 : \{\ell_1\}$ $(\text{refl}),5$
8. $\ell_2 : p$ $(\exists),2$
9. $\ell_0 : \exists s.\{\ell_2\}$ $(\exists),2$
10. $\ell_2 : \{\ell_2\}$ $(\text{refl}),8$
11. $\ell_3 : \neg\exists t.\exists s^{-1}.(p \sqcup \neg p)$ $(\exists),3$
12. $\ell_0 : \exists t.\{\ell_3\}$ $(\exists),3$
13. $\ell_3 : \neg\exists\neg t.\neg\exists s^{-1}.(p \sqcup \neg p)$ $(\neg\exists),4,12$
14. ▶ $\ell_3 : \exists t.\{\ell_2\}$ $(\neg\exists\neg),13,10$
15. $\ell_2 : \neg\exists s^{-1}.(p \sqcup \neg p)$ $(\neg\exists),11,14$
16. $\ell_0 : \neg(p \sqcup \neg p)$ $(\neg\exists^{-1}),15,9$
17. Unsatisfiable after a few steps
18. ▶ $\ell_2 : \neg\neg\exists s^{-1}.(p \sqcup \neg p)$ $(\neg\exists\neg),13,10$
19. $\ell_2 : \exists s^{-1}.(p \sqcup \neg p)$ $(\neg\neg),18$
20. ▶ $\ell_3 : \exists t.\{\ell_1\}$ $(\neg\exists\neg),13,7$
21. $\ell_1 : \neg\exists s^{-1}.(p \sqcup \neg p)$ $(\neg\exists),11,20$
-
22. ▶ $\ell_1 : \neg\neg\exists s^{-1}.(p \sqcup \neg p)$ $(\neg\exists\neg),13,7$
-

Figure 4: Global effect of the introduction of a new individual

1. $\ell_0 : \neg(\exists(s \sqcup \neg s). \neg \exists r.p \sqcup \neg \exists t. \neg \exists r.p)$ given
2. $\ell_0 : \{\ell_0\}$ (refl),1
3. $\ell_0 : \neg \exists(s \sqcup \neg s). \neg \exists r.p$ ($\neg \sqcup$),1
4. $\ell_0 : \neg \neg \exists t. \neg \exists r.p$ ($\neg \sqcup$),1
5. $\ell_0 : \exists t. \neg \exists r.p$ ($\neg \neg$),4
6. $\ell_0 : \neg \exists s. \neg \exists r.p$ ($\neg \exists \sqcup$),3
7. $\ell_0 : \neg \exists \neg s. \neg \exists r.p$ ($\neg \exists \sqcup$),3
8. ▶ $\ell_0 : \exists s. \{\ell_0\}$ ($\neg \exists \neg$),7,2
9. $\ell_0 : \neg \neg \exists r.p$ ($\neg \exists$),8,6
10. $\ell_0 : \exists r.p$ ($\neg \neg$),9
11. $\ell_1 : p$ (\exists),10
12. $\ell_0 : \exists r. \{\ell_1\}$ (\exists),10
13. $\ell_1 : \{\ell_1\}$ (refl),11
14. ▶ $\ell_0 : \exists s. \{\ell_1\}$ ($\neg \exists \neg$),7,13
15. $\ell_1 : \neg \neg \exists r.p$ ($\neg \exists$),14,6
16. $\ell_1 : \exists r.p$ ($\neg \neg$),15
17. ▶ $\ell_0 : \{\ell_1\}$ (ub),2,13
18. $\ell_2 : \neg \exists r.p$ (\exists),5
19. $\ell_0 : \exists t. \{\ell_2\}$ (\exists),5
20. $\ell_2 : \{\ell_2\}$ (refl),18
21. ▶ $\ell_0 : \exists s. \{\ell_2\}$ ($\neg \exists \neg$),7,20
22. $\ell_2 : \neg \neg \exists r.p$ ($\neg \exists$),6,21
23. Unsatisfiable (\perp),18,22
24. ▶ $\ell_2 : \neg \neg \exists r.p$ ($\neg \exists \neg$),7,20
25. Unsatisfiable (\perp),18,24
26. ▶ $\ell_0 : \neg \{\ell_1\}$ (ub),2,13
27. $\ell_2 : p$ (\exists),16
28. $\ell_1 : \exists r. \{\ell_2\}$ (\exists),16
29. $\ell_2 : \{\ell_2\}$ (refl),27
30. ▶ $\ell_0 : \exists s. \{\ell_2\}$ ($\neg \exists \neg$),7,29
31. $\ell_2 : \neg \neg \exists r.p$ ($\neg \exists$),30,6
32. $\ell_2 : \exists r.p$ ($\neg \neg$),31
- ... **Non-terminating** Repetition of 16–32
33. ▶ $\ell_2 : \neg \neg \exists r.p$ ($\neg \exists \neg$),7,29
34. Similarly to 30–32
35. ▶ $\ell_1 : \neg \neg \exists r.p$ ($\neg \exists \neg$),7,13
36. Similarly to 14–34
37. ▶ $\ell_0 : \neg \neg \exists r.p$ ($\neg \exists \neg$),7,2
38. Similarly to 8–36

Figure 6: An infinite derivation, due to unfair selection of concepts

1. $\ell_0 : \neg\exists(s \sqcup \neg s).\neg\exists r.p$ given
2. $\ell_0 : \{\ell_0\}$ (refl),1
3. $\ell_0 : \neg\exists s.\neg\exists r.p$ ($\neg\exists\sqcup$),1
4. $\ell_0 : \neg\exists\neg s.\neg\exists r.p$ ($\neg\exists\sqcup$),1
5. ▶ $\ell_0 : \neg\neg\exists r.p$ ($\neg\exists\neg$),4,2
6. $\ell_0 : \exists r.p$ ($\neg\neg$),5
7. $\ell_1 : p$ (\exists),6
8. $\ell_0 : \exists r.\{\ell_1\}$ (\exists),6
9. $\ell_1 : \{\ell_1\}$ (refl),7
10. ▶ $\ell_1 : \neg\neg\exists r.p$ ($\neg\exists\neg$),4,9
11. $\ell_1 : \exists r.p$ ($\neg\neg$),5
12. ▶ $\ell_0 : \neg\{\ell_1\}$ (ub),2,9
13. $\ell_2 : p$ (\exists),11
14. $\ell_1 : \exists r.\{\ell_2\}$ (\exists),11
- ... **Non-terminating** Repetition of 7–14
15. ▶ $\ell_0 : \{\ell_1\}$ (ub),2,9
- Never expanded
16. ▶ $\ell_0 : \exists s.\{\ell_1\}$ ($\neg\exists\neg$),4,9
- Never expanded
17. ▶ $\ell_0 : \exists s.\{\ell_0\}$ ($\neg\exists\neg$),4,2
- Never expanded

Figure 7: An infinite derivation, due to unfair selection of branches